Summer Project

School of Mathematics, Statistics & Physics

Summer 2018

Workbook 1

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Introduction

This project is intended to give current A-Level students (or equivalent) an opportunity to demonstrate their mathematical knowledge and experience, and to give such students a taste of some new material which most will not have seen before. A brief recap of some concepts encountered in the A-level syllabus is provided, with a more detailed description of new concepts to provide assistance when preparing solutions to questions posed in the project. This project consists of three components:

- A written assignment sheet
- Some computer based questions using our NUMBAS online testing system
- A more open ended, research-style question

The questions are designed to be of varying difficulty to challenge students at different skill levels and we would expect a good student to take up to five days to complete all components. Any questions marked with a " * " will involve concepts beyond the standard A-level syllabus and we strongly recommend reading through the information in this booklet carefully before attempting the questions. Completed assignments will

be graded and successful students will be awarded with a certificate and general feedback. More details about this summer project can be found on our dedicated webpage: `http://www.mas.ncl.ac.uk/~nlf8/summer-project/`.

Written assignment

There are nine questions to work through here. Most of these questions will be based on material covered in this booklet. However, there may be one or two surprises! Solutions to these questions can be hand-written or typed. Hand-written solutions should be scanned and emailed to `maths.physics@ncl.ac.uk`. Solutions prepared in a word-processing package (eg. Microsoft Word) can be emailed directly, although we recommend exporting to portable document format (PDF) first, to help prevent loss of formatting. Students must submit their solutions by **4pm, Friday 31st August GMT**.

Computer based test

This uses Newcastle University's *NUMBAS online testing system*. The test will be available in two "modes"; *PRACTICE* and *ASSESSED*. Students can access the practice test as many times as they like. Each time they open a test, the format of the questions within that particular test will remain the same, but the numbers might change. The assessed mode will be active for two weeks only, from **Monday 20th August–Friday 31st August**. Students must login and complete the test by **4pm, Friday 31st August GMT**. Only **one** attempt will be allowed, but students can pause the test as often as they wish – there are no time constraints. The practice mode will be available throughout August and September. Students are strongly advised to work through our *Introduction to NUMBAS* first – available at
`https://numbas.mathcentre.ac.uk/test-yourself/getting-started/`. A link to the actual test, and further instructions about submission of the test will be available on our webpage soon.

Research-style question

This is the final part of the project. Students will be given a choice of three questions, from which they are required to attempt only **one**. We expect solutions to any of these questions to be one or two pages long. Students can pick from a range of topics for this question, including options from statistics or number theory. Most questions do not have strict guidelines and we encourage students to be creative with their answers. As with the written assignment, solutions to the open-ended questions can be either hand-written or electronically produced with a word processing package. Answers should be emailed to `maths.physics@ncl.ac.uk` no later than **4pm, Friday 31st August GMT**.

1 Some revision

1.1 Quadratic equations

A root of the function $f(x)$ is any value of x which solves $f(x) = 0$. For a quadratic polynomial where $f(x) = ax^2 + bx + c$, there is a maximum of two roots. These can be found via three different methods:

1. Factorisation:

   The quadratic equation $ax^2 + bx + c = 0$ can sometimes be expressed in the form $(dx + e)(jx + k) = 0$.

   This implies that the roots are $x = -\frac{e}{d}$ and $x = -\frac{k}{j}$.

2. Quadratic formula:

   Substitute the respective values of a, b and c into the quadratic formula:

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Completing the square (only when $a = 1$):

   $x^2 + bx + c = (x + b/2)^2 - (b/2)^2 + c = 0 \Rightarrow (x + b/2)^2 = (b/2)^2 - c.$

1.2 Plane geometry

The equation of a straight line in *gradient form* is

$$y = mx + c,$$

where m is the *gradient* and c is the *y-intercept*. The equation of the straight line between points (x_1, y_1) and (x_2, y_2) in the xy-plane can be found using the formulae:

$$y - y_1 = m(x - x_1), \quad \text{where} \quad m = \frac{y_1 - y_2}{x_1 - x_2}.$$

Points of intersection between two functions can be identified by equating them and finding values of x which satisfy the relationship. Some pairs of functions may have many intersection points and others may not have any.

1.3 Binomial expansion

An expression of the form $(a+b)^n$ can be written out as the complete sum of all the individual terms using the binomial formula given below:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where $\binom{n}{k}$ is the binomial coefficient which can be obtained using a calculator or by writing out Pascal's triangle. The binomial coefficient can also be calculated as a quotient of factorials given in the formula below:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

1.4 Logarithms and exponentials

Index form: $y = a^x$ *Logarithmic form:* $\log_a(y) = x$

Here is a list of useful logarithmic identities:

- $\log_a(x) + \log_a(y) = \log_a(xy)$
- $\log_a(x) - \log_a(y) = \log_a(\frac{x}{y})$
- $\log_a(x^k) = k\log_a(x)$

The exponential function $f(x) = e^x$ is a special function because $\dfrac{df}{dx} = f(x)$. The natural logarithm is $\log_e$ and is often denoted ln. If at any point in the exercises you encounter a log without a subscript then it can be assumed it means $\log_e$, not $\log_{10}$ as is sometimes the convention.

1.5 Trigonometric functions

You should all be familiar with $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ and how they link to the ratio of lengths in a right-angled triangle. Radians are a measure of angle where 2π radians $= 360°$. One can assume that the angle is given in radians unless explicitly expressed otherwise. Some important trigonometric identities are:

- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $sin(\theta) = a \quad \Rightarrow \quad \theta = sin^{-1}(a)$
- $cos(\theta) = b \quad \Rightarrow \quad \theta = cos^{-1}(b)$

1.6 Differentiation and integration

For a polynomial function $f(x) = ax^n$, the *first-order derivative* $\dfrac{df}{dx} = f'(x) = anx^{n-1}$. The inverse of this operation is *integration*, where for $f(x) = ax^n$ then

$$\int f(x)\ dx = \frac{a}{n+1}x^{n+1} + c,$$

where c is the constant of integration.

Differentiation can be used to find the gradient at any point of a particular function. Substituting any given x value in to the derivative function will give the gradient of $f(x)$ at that point. Points where the gradient equals zero are called turning points and can be important when trying to find values of x which maximize or minimize a function.

Integration can be used to calculate the area between a function and the x axis. If we want to calculate the area under the curve between two values of x we select the lowest value of x as our lower limit of integration, l, and the higher value as our upper limit of integration, u. Mathematically this can be written as:

$$\int_l^u f(x)\ dx = \left[\int f(x)\ dx\right]_l^u = \int f(x)\ dx\Big|_{x=u} - \int f(x)\ dx\Big|_{x=l},$$

where the notation $|_{x=u}$, for example, means "evaluated at $x = u$". One important thing to consider is that the area below the x-axis will have a negative contribution to the overall integral so it may be necessary to calculate the area above and below the x-axis separately and then add them together.

Here is a table of useful known derivatives and integrals:

Integral	Function	Derivative
$-\frac{1}{k}\cos(k\theta)$	$\sin(k\theta)$	$k\cos(k\theta)$
$\frac{1}{k}\sin(k\theta)$	$\cos(k\theta)$	$-k\sin(k\theta)$
$\frac{1}{k}e^{kx}$	e^{kx}	ke^{kx}
$x\ln(x) - x$	$\ln(x)$	$\frac{1}{x}$

1.6.1 Differentiation rules

The chain rule: The chain rule is used when differentiating a function of a function. So for a function $y(x) = f(g(x))$, we have:

$$\frac{dy}{dx} = \frac{dy}{dg} \times \frac{dg}{dx}.$$

The product rule: The product rule is used when differentiating a product of two functions. For a function of the form $y(x) = f(x)g(x)$, then:

$$\frac{dy}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}, \quad \text{alternatively written as} \quad \frac{dy}{dx} = f'g + g'f,$$

where the $'$ denotes the derivative with respect to x.

The quotient rule: The quotient rule is used when differentiating a function divided by a function. For a function in the form $y(x) = \dfrac{u}{v}$, then

$$\frac{dy}{dx} = \frac{u'v - v'u}{v^2},$$

again where the $'$ denotes a derivative with respect to x.

1.6.2 Integration rules

Integration by substitution: When integrating a function of the form

$$\int f(g(x))g'(x)\ dx,$$

we substitute $g(x) = u$ and $du = g'(x)dx$, which gives us

$$\int f(u)du.$$

Integration is then carried out normally with respect to u, before reverting back to x, as in the original function.

Integration by parts: Integration by parts is used when integrating a product of two functions. For $f(x) = u(x)v(x)$, the formula is:

$$\int uv\ dx = u\int v\ dx - \int u'\left(\int v\ dx\right) dx.$$

2 New topic 1: Partial differentiation

We expect most students not to have seen this topic before. Calculus of multivariate functions is an extremely broad subject, filled with complicated proofs and mathematical techniques with huge applications in the real world. We aim to introduce the basic concepts followed by some simple examples. The purpose of including this new section is to give students a chance to familiarize themselves with certain mathematical computations which will be required of them should they pursue mathematics beyond A-level.

2.1 Multivariate functions

We are all familiar with functions of one variable, often described as "y as a function of x". What if we consider a function of more than one variable? Frequently in the real world there may be more than one variable affecting the overall change of an output. For example, both age and height could contribute to a change in weight (amongst many other factors). We could choose to model this relationship using a function of both age and height, eg:

$$\text{weight} = c_1 \times \text{age} + c_2 \times \text{height},$$

where c_1 and c_2 are coefficients for age and height respectively. This principle can be applied more generally. Here are examples of functions of more than one variable, so-called *multivariate functions*:

- $f(x, y) = ax^2 + \sin(y)$
- $f(x_1, x_2, ..., x_n) = a_n x_n + ... + a_1 x_1 + a_0$

In these examples, y is **not** a function of x and this will remain the case for the remainder of the following sections. Functions of two variables can be plotted in 3D to produce all kinds of weird and wonderful shapes. For example, $f(x, y) = \pm\sqrt{1^2 - x^2 - y^2}$ is a function for the surface of a sphere of radius 1 centered at the origin. Plotting functions of two variables in three dimensions has a huge number of real world applications. However, for this project we will only be introducing you to the algebra/calculus of multivariate functions and we will not go into any more depth about their geometry.

2.2 Partial differentiation

You may be asking the question "how do we calculate the rate of change of any one variable in a multivariate function?". You will be pleased to hear this is a surprisingly simple procedure!

Important result: to calculate the derivative with respect to one variable in a multivariate function we simply differentiate the function with respect to our chosen variable and treat all other variables as constants. This is known as *partial differentiation*.

Because we only partially differentiate the function when we differentiate with respect to one variable, the resulting derivative is called the *partial derivative*. The partial derivative is denoted slightly differently. For a function $f = f(x, y)$ the partial derivative of f with respect to x is denoted $\frac{\partial f}{\partial x}$. Notice here we use "$\partial$" rather than "$d$".

Let us consider some examples.

Example 1: Find $\frac{\partial f}{\partial x}$ of $f(x, y) = x^2 + 2xy$.

Solution:

$$\begin{aligned} f(x, y) &= x^2 + 2xy, \\ \frac{\partial}{\partial x}(x^2) &= 2x, \\ \frac{\partial}{\partial x}(2xy) &= \frac{\partial}{\partial x}([2y]x), \\ &= 2y. \end{aligned}$$

Because here we treat y as if it were a constant and therefore, $2y$ as the coefficient of x.

$$\Rightarrow \quad \frac{\partial f}{\partial x} = 2x + 2y,$$

Now let us find the rate of change of the same function with respect to y.

$$\begin{aligned} f(x, y) &= x^2 + 2xy, \\ \frac{\partial}{\partial y}(x^2) &= 0. \end{aligned}$$

The x^2 term has no y dependence and so it differentiates to 0.

$$\begin{aligned} \frac{\partial}{\partial y}(2xy) &= \frac{\partial}{\partial y}([2x]y) \\ &= 2x, \end{aligned}$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = 2x.$$

Example 2: Find the derivatives with respect to x and y of $f(x, y) = x\sin(y)$.

Solution:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x\sin(y)),\\ &= \frac{\partial}{\partial x}([\sin(y)]x),\\ &= \sin(y).\end{aligned}$$

Here we treat $\sin(y)$ as the coefficient of x. Similarly for the derivative with respect to y we get:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x\sin(y)\right),\\ &= \frac{\partial}{\partial y}\left([x]\sin(y)\right),\\ &= x\cos(y).\end{aligned}$$

Example 3: Find the derivatives with respect to x and y of $f(x, y) = 4x^3y^2 + e^{3xy}$.

Solution:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(4x^3y^2 + e^{3xy}),\\ &= \frac{\partial}{\partial x}([4y^2]x^3 + e^{[3y]x}),\\ &= 12x^2y^2 + 3ye^{3xy}.\end{aligned}$$

Similarly for the derivative with respect to y:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(4x^3y^2 + e^{3xy}),\\ &= \frac{\partial}{\partial x}([4x^3]y^2 + e^{[3x]y}),\\ &= 8x^3y + 3xe^{3xy}.\end{aligned}$$

(Note: all the usual rules of differentiation, such as the product rule etc, can be applied in the normal way.)

The partial derivative represents the rate of change of one variable, at any given point, if it is increased by an infinitesimally small amount and all other variables are kept constant. This is the same definition as a standard derivative in one variable just here we are ignoring the change brought about by any other variables by assuming they are constant.

2.3 Stationary points

You should all be familiar with turning points for functions of one variable. Turning points can be found when the gradient is equal to zero. Stationary points are the equivalent of turning points for functions of two variables drawn in three dimensions. A stationary point is a point where the rate of change of both x and y are zero simultaneously. In other words a stationary point is any point (x, y) where both

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0.$$

To find a stationary point we find the x and y coordinates by solving $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously. Here are some examples.

Example 4: Considering the same function as from **Example 1:** $f(x, y) = x^2 + 2xy$. Find all stationary points.

Solution:

$$\frac{\partial f}{\partial x} = 2x + 2y$$
$$\frac{\partial f}{\partial y} = 2x$$

So for,

$$\frac{\partial f}{\partial y} = 0 \;\Rightarrow\; 2x = 0 \;\Rightarrow\; x = 0$$
$$\frac{\partial f}{\partial x} = 0 \;\Rightarrow\; 2y = 0 \;\Rightarrow\; y = 0$$

So the point $(0, 0)$ is the only stationary point.

Example 5: Considering the same function as from **Example 2:** $f(x, y) = x\sin(y)$. Find all stationary points.

Solution:

$$\frac{\partial f}{\partial x} = \sin(y)$$
$$\frac{\partial f}{\partial y} = x\cos(y)$$

So for,

$$\frac{\partial f}{\partial x} = 0 \;\Rightarrow\; \sin(y) = 0 \;\Rightarrow\; y = 0, \pi, 2\pi, \cdots$$
$$\frac{\partial f}{\partial x} = 0 \;\Rightarrow\; x\cos(y) = 0$$

$\cos(y) \neq 0$ for $y = 0, \pi, 2\pi, \cdots$ so this implies that

$$x = 0$$

So the stationary points are $(0, n\pi)$ where $n = 0, 1, 2, \cdots$.

3 New topic 2: Double integration

Following on from section 2, how do we integrate a function of more than one variable? We do so in a similar way to partial differentiation, that is we integrate over one variable and treat the other as a constant. For a function of two variables between two limits we integrate twice, first with respect to one variable and substitute in the limits and then we integrate with respect to the other variable. This is called *double integration* and has all manner of applications, one being involved in calculating the area of a surface. Let us consider some examples.

Example 6: Calculate $I = \int_{y=0}^{1} \int_{x=0}^{1} 3x^2 + 2y \; dxdy$.

Solution:

$$\begin{aligned} &\text{Let } f(x) = 3x^2 + 2y, \\ &\int 3x^2 \; dx = x^3, \\ &\int 2y \; dx = 2xy. \end{aligned}$$

This is because here we treat y as if it were a constant.

$$\Rightarrow \quad \int f(x) \; dx = x^3 + 2xy.$$

The lower and upper limits for x are 0 and 1 respectively. Substituting this result into our integral I gives,

$$I = \int_{y=0}^{1} \left[x^3 + 2xy\right]_{x=0}^{1} dy.$$

Substituting the limits in for all occurrences of x gives,

$$\begin{aligned} I &= \int_{y=0}^{1} (1 + 2y) - (0 + 0) \; dy, \\ &= \int_{y=0}^{1} (1 + 2y) \; dy. \end{aligned}$$

Now we must integrate the resulting function with respect to y and substitute in the limits.

$$\begin{aligned} \Rightarrow \quad I &= \left[y + y^2\right]_0^1, \\ &= 2 - 0, \\ &= 2. \end{aligned}$$

Example 7: Calculate $I = \int_{x=0}^{1} \int_{y=0}^{\pi} x \sin(y) \; dydx$.

Solution:

$$\text{Let } f(y) = x\sin(y),$$
$$\int f(y) \; dy = -x\cos(y).$$

This is because here we treat x as if it were a constant. The lower and upper limits for y are 0 and π respectively. Substituting this result into our integral I gives,

$$I = \int_{x=0}^{1} [-x\cos(y)]_{y=0}^{\pi} \; dx.$$

Substituting the limits in for all occurrences of y gives,

$$\begin{aligned} I &= \int_{x=0}^{1} (-x \times \cos(\pi)) - (-x \times \cos(0)) \; dx, \\ &= \int_{x=0}^{1} (-x \times -1) - (-x \times 1) \; dx, \\ &= \int_{x=0}^{1} 2x \; dx. \end{aligned}$$

Now we must integrate the resulting function with respect to x and substitute in the limits.

$$\begin{aligned} I &= \left[x^2\right]_{x=0}^{1}, \\ &= 1 - 0, \\ &= 1. \end{aligned}$$

In these situations, where both the limits are numbers, the order of integration makes no difference, that is integrating over x or y first will arrive at the same answer. The reader is encouraged to do so to verify this result for themselves.

To make things more complicated the limits could contain one of the variables and then the order of integration does make a difference. The following is an example of a situation where the order of integration does make a difference.

Example 8: Calculate $I = \int_{x=0}^{\pi} \int_{y=0}^{x} 6x + \cos(y)\ dydx$.

Solution: It is important here that we perform the y integral first because the y limit has x dependence.

$$\begin{aligned} \text{Let } f(y) &= 6x + \cos(y) \\ \int f(y)\ dy &= 6xy + \sin(y) \\ \Rightarrow \quad I &= \int_{x=0}^{\pi} [6xy + \sin(y)]_{y=0}^{x}\, dx \end{aligned}$$

Because the upper limit of y is equal to x we have to replace all occurrences of y with x and then subtract 0 substituted in for y. This gives,

$$\begin{aligned} I &= \int_{x=0}^{\pi} \left(6x^2 + \sin(x)\right) - (0 + 0)\ dx \\ &= \int_{x=0}^{\pi} \left(6x^2 + \sin(x)\right)\ dx \\ &= \left[2x^3 - \cos(x)\right]_0^{\pi} \\ &= (2\pi^2 - \cos(\pi)) - (0 - \cos(0)) \\ &= (2\pi^3 + 1) - (-1) \\ &= 2\pi^3 + 2 \end{aligned}$$

4 Written assignment

Questions marked with a " * " will involve concepts beyond the standard A-level syllabus and we strongly recommend reading through the attached notes carefully before attempting these questions.

Question 1

Find the points of intersection between $f(x) = \dfrac{2x+1}{x+2}$ and $g(x) = \dfrac{x-3}{x-6}$.

[4 marks]

Question 2

Find a value of θ which satisfies the following expression:

$$cos^2\left(2\theta - \frac{\pi}{4}\right) - 1 = 0, \qquad 0 < \theta < 2\pi.$$

[4 marks]

Question 3

Given that $x = a^n$, show that $\log_a(x^k) = k\log_a(x)$.

[4 marks]

Question 4*

Find the co-ordinates of the stationary point of $f(x, y) = 4x^2 - 2xy + 2y$.

[5 marks]

Question 5*

The number, in thousands, of reactions in one second of a particular chemical reaction (denoted n), could be explained as a relationship of both temperature (t) and percentage concentration of chemical (c). This relationship could be written as the following function.

$$n = -t^2 + 50t + 10 + 50ln(c) - c$$

(a) Write a function for the rate of change in number of reactions if we change only temperature.

(b) Write a function for the rate of change in number of reactions if we change only chemical concentration.

(c) Find the values of temperature and chemical concentration that maximise the number of reactions.

[6 marks]

Question 6*

Perform double integration on $f(x, y)$ between the limits given below:

$$f(x, y) = x^2 + y^2, \quad 0 < x < 1, \quad 0 < y < 1.$$

[5 marks]

Question 7*

Evaluate:

$$\int_{x=0}^{2} \int_{y=0}^{x} 3y^2 + \frac{1}{x} e^x \, dydx.$$

[6 marks]

Question 8

Archimedes lived one quarter of his life as a boy, one fifth as a youth, one third as a man and lived thirteen years in his old age. How old was Archimedes?

[5 marks]

Question 9

Figure 1 shows an equilateral triangle with sides of length 24cm. Find the shaded area.

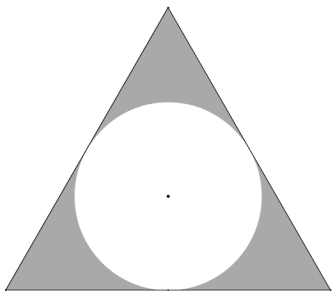

Figure 1

[5 marks]

5 Research question

The final part of the project is a more open ended research based question. We expect a one to two page report presenting all the information related to the research question.

Choose **one** of the following research topics,

[20 marks]

Maclaurin series

One surprising result is that useful functions such as $\sin(x)$ and e^x can be written as a polynomial up to an infinite power of x. These are known as Taylor or Maclaurin series. Find the Maclaurin series for some important functions and show how they can be calculated. Also verify the series representations with some basic examples and plots.

Chicken egg hatching probability

A farmer is investigating the success probability of his chickens at hatching healthy chicks. He has 1001 eggs and 644 hatch. What would be an appropriate probability distribution to use to model the data. Calculate some important data summaries (eg expectation, standard deviation).

He sells each chicken for £3. What is the probability that in the next hatching season he can make £2000 (assuming that the same number of eggs are laid and all other conditions are the same)?

A different breed of chicken lays eggs with a hatching success probability of 0.692. Is there a statistically significant difference between the hatching probabilities of the two breeds? Justify all your conclusions. You are given that the standard error for the hatching probability is 0.01514.

Divergent infinite series

Some infinte series have caused heated debate between mathematicians for centuries. One famous example is Grandi's series $1-1+1-1+1-1+1-\cdots$ it can be shown mathematically that this sums to 3 different values: $0, 1$ and most importantly for this research question $1/2$. Show how Grandi's series can sum to $1/2$ and then use this result to prove a frankly unbelievable result, that $1 + 2 + 3 + 4 + \cdots = -1/12$. Comment on the validity of the proof.