Probability and Probability Models

Dr. Lee Fawcett

Newcastle University

Introduction

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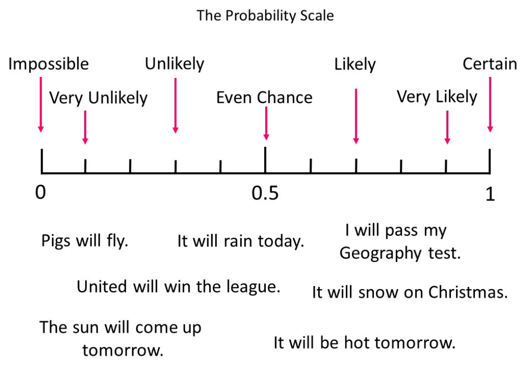
We're probably also used to quantifying likelihood using percentages:

- "England have a 20% chance of winning the World Cup"
- "I'm 99% sure I turned the oven off before I left for the airport..."

What is Probability?



The Probability Scale

Pigs will fly.

It will rain today.

I will pass my Geography test.

United will win the league.

It will snow on Christmas.

The sun will come up tomorrow.

It will be hot tomorrow.

Defining Probability

We can set-up a mathematically rigorous framework to define probability; we can then use this to calculate the probability of whether something we're interested in will happen.

First we need a few definitions:

- **Experiment**: Something that generates data
- **Outcome**: A possible result from the experiment
- **Event**: A collection of one or more outcomes

Classical Probability

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For example:

$$P(\text{A playing card is a 9}) = \frac{4}{52} = \frac{1}{13} = 7.69\%$$

Estimating Probabilities

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$$P(\text{Event } A) = \frac{\text{Number of times event } A \text{ occurs}}{\text{Number of times the experiment was run}}$$

Random Variables

Random variables are outcomes of interest that will vary randomly between runs of an experiment. For example:

- The temperature outside this building each day
- The number of unique visitors to a website every hour
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We are typically interested in the probability a random variable takes a particular value (or range of values).

Random Variables

Suppose we are interested in the probability of a fair, six-sided die taking different values.

We can easily write down the **probability mass function** (PMF) for X : Outcome of roll of die:

x	1	2	3	4	5	6
$P(X = x)$	1/6	1/6	1/6	1/6	1/6	1/6

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```
> N = 100000
> game = sample(1:6, N, replace=TRUE)
> table(game)
game
    1     2     3     4     5     6
16473 16935 16677 16570 16504 16841
> table(game)/N
game
      1       2       3       4       5       6
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  - $P(\text{Two sixes}) = (1/6) \times (1/6) = 1/36$

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- Also:
  - **Valid PMF**: All probabilities sum to 1
  - **Complement**: $P(X = 2) + P(X \neq 2) = 1/6 + 5/6 = 1$

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- X: Number of tropical storms in any given year for a location in the Gulf of Mexico
- X: Number of rats developing renal tumours after expore to *Ochratoxin A*

Introduction

We will now break for a problem session covering this section.

Please pause the video here and go to **Problem Session 1** of **Lesson 4**.

See you in a bit!

Discrete Probability Models

Introduction

The table below shows the number of years, in the past 100 years, in which X tropical storms were observed for a location in the Gulf of Mexico.

x	0	1	2	3	4	5	6	7+
# Years	14	27	27	18	9	4	1	0

All things being equal, can we use this information to assess the likelihood of tropical storms at this location in future years?

Introduction

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$$P(X \leq 3) = \frac{14 + 27 + 27 + 18}{100} = \frac{86}{100} = 0.86.$$

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This simple approach to probability would give us

$$P(X > 6) = \frac{0}{100} = 0,$$

i.e. *impossible*! Does this make sense?

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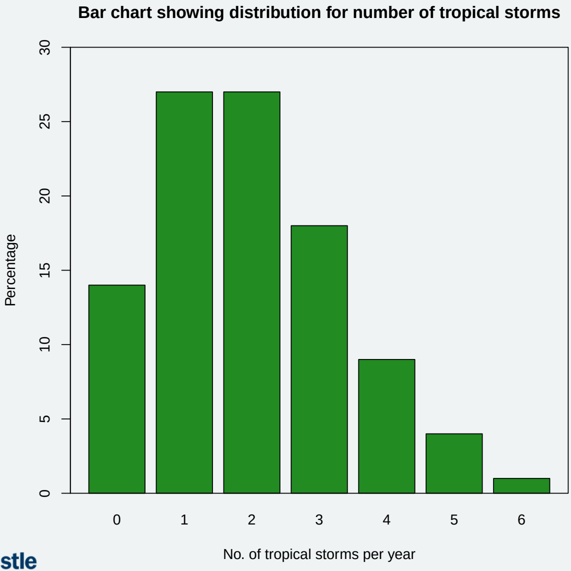
- These probabilities are **frequency estimates** of the true probabilities
- What are the true probabilities? Difficult to know, as we can't assume equally likely outcomes
- We might consider the PMF/ as an approximation to the 'true' underlying probability distribution, governed by a **probability model**

Introduction



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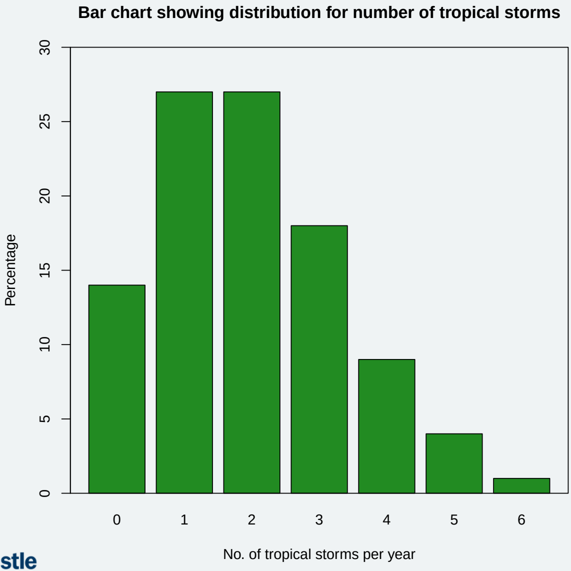

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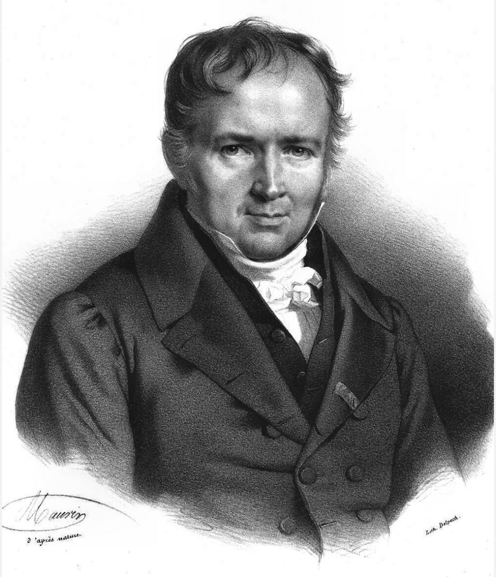
Assuming ‘Yes’, ‘Yes’ and ‘No’ gives rise to the **Poisson distribution** as a suitable probability model for X.

The Poisson distribution


Maurin
d'après nature.
Lith. Delpech.

The Poisson distribution: Notation and formulae

We say $X \sim Po(\lambda)$, with

$$P(X = x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!},$$

where $\lambda > 0$, $x = 0, 1, \ldots$ and $x! = x \cdot (x-1) \cdot (x-2) \ldots$.

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We also have

- Exepcted value $= E(X) = \lambda$
- Variability $= Var(X) = \lambda$

The Poisson distribution: Notation and formulae

For example, in our tropical storms dataset we have

$$\lambda = E(X) \approx \bar{x} = \frac{1 \cdot 27 + 2 \cdot 27 + \cdots + 6 \cdot 1}{100} = 1.97;$$

in other words, we assume a constant rate of around 1.97 tropical storms per year. Then

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$$P(X = 3) = \frac{e^{-1.97} 1.97^3}{3!} = 0.178 \quad \textit{(Compared to 0.18 earlier)}$$

Similarly,

$$P(X = 6) = \frac{e^{-1.97} 1.97^6}{6!} = 0.011 \quad \textit{(Compared to 0.01 earlier)}$$

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For the full probability distribution, we have:

x	0	1	2	3	4	5	6	7+
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Empirical	0.14	0.27	0.27	0.18	0.09	0.04	0.01	0.00

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Notice:

- How close the Poisson probabilities are to the **empirical** probabilities
- That although $P(X > 6)$ is very small, at least we don't rule this out now!

The Binomial distribution

In a study to investigate mammalian toxicity to *Ochratoxin A* (OTA), five male rats were exposed to OTA. Subsequently, the number of rats developing renal tumours was recorded.

Previous studies (e.g. Herman, D. and Mantle, P., 2021) estimate that 35% of male rates exposed to OTA will go on to develop renal tumours.

The Binomial distribution

If we are interested in X: The number of rats that will develop tumours, we might assume a **Binomial distribution**:

- We have a fixed number of ‘trials’ (call this n, with $n = 5$ in this example), meaning there is a fixed upper limit to the value X can take

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then we might say $X \sim Bin(n = 5, p = 0.35)$.

The Binomial distribution: Notation and formulae

We say $X \sim Bin(n, p)$, with

$$P(X = x) = \left(\begin{array}{c} n \\ x \end{array} \right) p^x (1 - p)^{n-x},$$

where $n = 1, 2, 3, ...,$ $0 \leq p \leq 1$ and $x = 0, 1, \ldots$.

The Binomial distribution: Notation and formulae

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$$P(X = x) = \left(\begin{array}{c} n \\ x \end{array} \right) p^x (1 - p)^{n-x},$$

where $n = 1, 2, 3, ...,$ $0 \leq p \leq 1$ and $x = 0, 1, \ldots$.

We also have

- $E(X) = np$
- $Var(X) = np(1 - p)$

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- $E(X) = np$
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This gives the following PMF for $X \sim Bin(5, 0.35)$:

x	0	1	2	3	4	5
$P(X = x)$	0.116	0.312	0.336	0.181	0.049	0.005

Discrete Probability Models

We will now break for a problem session covering this section.

Please pause the video here and go to **Problem Session 2** of **Lesson 4**.

See you in a bit!

The Normal distribution

Studying Normal populations

It is often the case that collected data have a distribution with the characteristic shape of the **Normal distribution**.

Studying Normal populations

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To explain this phenomenon, let's consider an example.

Haematocrit (Packed Cell Volume)

The **haematocrit** (Ht or HCT) or **packed cell volume** (PCV) is a measurement of the proportion of blood volume that is occupied by red blood cells.

It is normally 45 ± 7 $(38\% - 52\%)$ for males and 42 ± 5 $(37\% - 47\%)$ for females.

Haematocrit (Packed Cell Volume)

In a recent investigation, a haematocrit measurement was taken from 126 female students at Newcastle University:

42.0	42.0	40.0	45.0	42.0	43.0	36.0	43.0	40.0	42.0	35.0
46.0	45.0	38.0	40.0	46.0	45.0	43.0	42.0	41.0	44.0	41.0
49.0	42.0	35.0	46.0	44.0	37.0	43.0	42.0	41.0	36.0	41.0
44.0	38.0	40.5	42.0	38.0	35.0	42.0	42.0	40.0	45.0	40.0
55.0	45.0	44.0	44.0	43.0	42.0	42.0	43.0	42.0	45.0	43.0
44.0	41.0	46.0	49.0	44.0	44.0	32.5	42.0	44.0	49.0	41.5
48.0	42.0	39.0	45.0	41.0	42.0	43.0	46.0	40.0	43.0	42.0
44.0	40.0	44.0	41.0	44.0	43.0	49.0	40.5	39.0	48.0	40.0
41.0	41.0	41.0	45.0	36.0	39.0	36.0	38.0	45.0	46.0	41.0
40.0	43.0	34.5	42.0	42.0	39.0	41.5	46.0	42.0	44.0	46.0
44.0	40.0	38.5	40.0	40.0	44.0	39.0	40.0	39.0	43.0	42.0
36.0	46.0	44.5	48.0	45.5						

Haematocrit (Packed Cell Volume)

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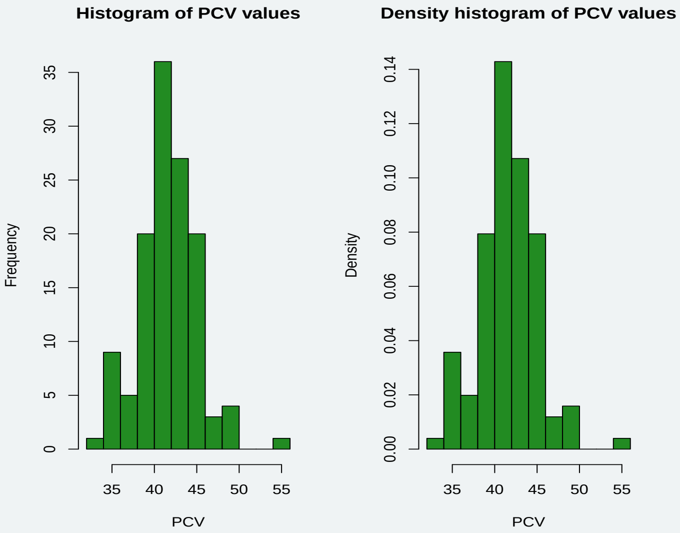
We can use techniques from Lecture 1 to summarise these data:

- We could summarise the measurements **numerically**
  - the **mean** haematocrit measurement is 42.15
  - the **standard deviation** is 3.44
- We could summarise the measurements **graphically** – perhaps using a **histogram**

Haematocrit (Packed Cell Volume)



Haematocrit (Packed Cell Volume)

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It is characterised by the symmetrical **bell-shaped curve**:

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It is characterised by the symmetrical **bell-shaped curve**:

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- values further away *tail off* in terms of their frequencies

A ‘perfect’ Normal distribution curve is given by the function

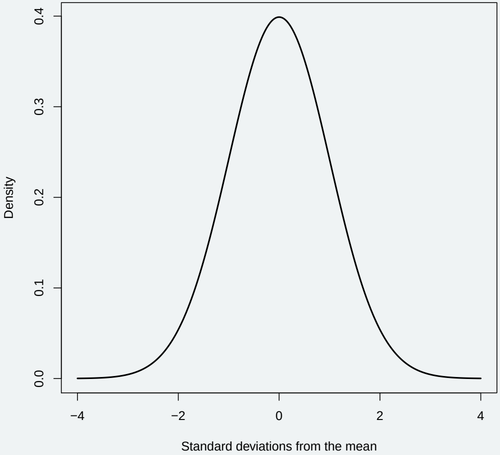
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right\}.$$

This is known as the **probability density function** (PDF) of the Normal distribution, where μ and σ represent the mean and standard deviation (respectively).

The Normal (Gaussian) distribution

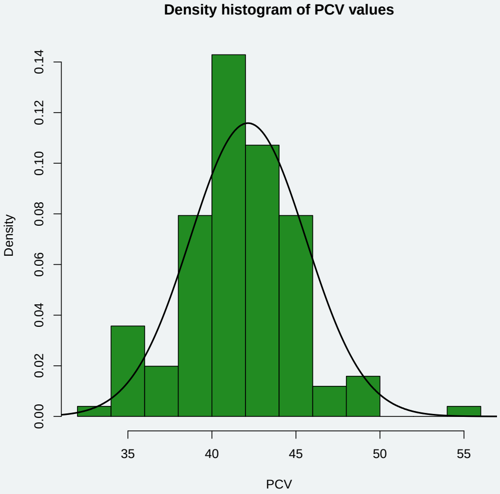
Haematocrit (Packed Cell Volume)

The plot below shows the PDF of the ‘perfect’ Normal distribution:



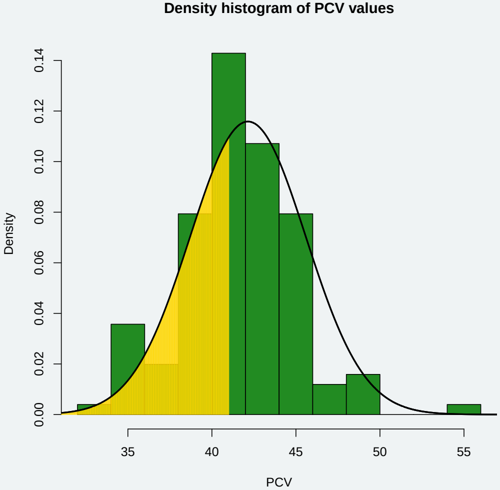
Haematocrit (Packed Cell Volume)

The Normal distribution curve for our haematocrit data, using $\mu = 42.15$ and $\sigma = 3.44$, looks like:



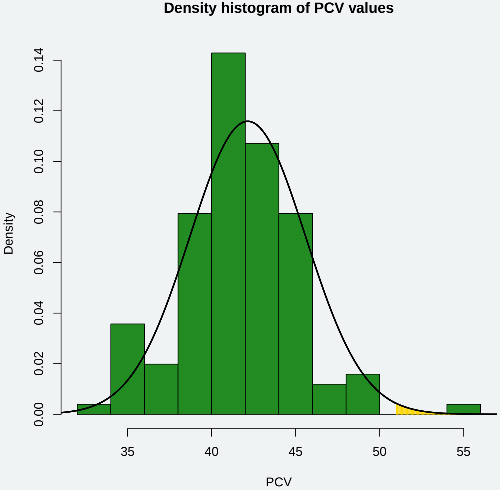
Haematocrit (Packed Cell Volume)

Areas under the curve represent probabilities; for example, for $P(X < 41)$:

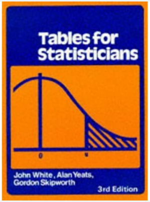


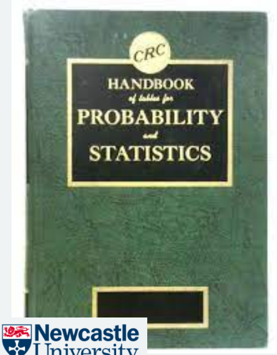
Haematocrit (Packed Cell Volume)

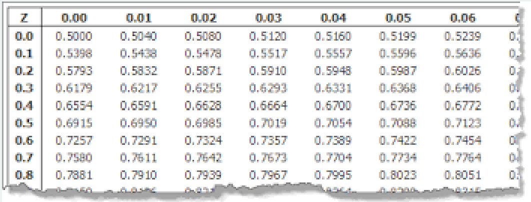
Areas under the curve represent probabilities; for example, for $P(X > 51)$:



Haematocrit (Packed Cell Volume)







Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051

Normal Q-Q plots

In the Haematocrit example, the sample size was sufficiently large to compare a histogram with the fitted PDF of a Normal distribution.

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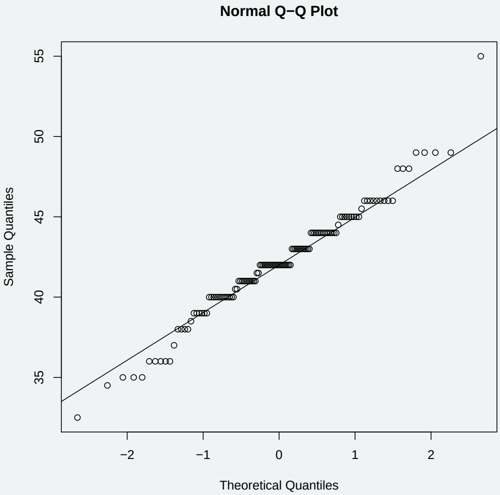
With smaller samples, assessing Normality in such a way can often be challenging.

Normal quantile plots (often referred to as ‘Q-Q plots’) are more often used to assess Normality.

Such plots provide the main way of checking the assumption of Normality underpinning much of what we will cover in tomorrow’s sessions.

Normal Q-Q plots



The Normal distribution

We will now break for a problem session covering this section.

Please pause the video here and go to **Problem Session 3** of **Lesson 4**.

See you in a bit!

Conclusions

1. **Basic ideas of probability** – probability scale, language/notation, classical and frequency calculations
2. **Probability models**
    - For discrete random variables: The Poisson and binomial distributions
    - For Continuous random variables: The Normal distribution
3. Checking **Normality**