

The development of the theory of automatic groups

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Abstract

We describe the development of the theory of automatic groups. We begin with a historical introduction, define the concepts of automatic, biautomatic and combable groups, derive basic properties, then explain how hyperbolic groups and the groups of compact 3-manifolds based on six of Thurston's eight geometries can be proved automatic. We describe software developed in Warwick to compute automatic structures, as well as the development of practical algorithms that use those structures. We explain how actions of groups on spaces displaying various notions of negative curvature can be used to prove automaticity or biautomaticity, and show how these results have been used to derive these properties for groups in some infinite families (braid groups, mapping class groups, families of Artin groups, and Coxeter groups). Throughout the text we flag up open problems as well as problems that remained open for some time but have now been resolved.

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1 Introduction

This chapter describes the development of the theory of automatic groups. It aims to explain the definition, and put that into mathematical and historical context, to detail what is known, give brief accounts of some of the big problems in the subject that have already been solved, and describe those problems that remain open.

Thurston is credited with the definition of automatic groups, and is one of six authors of one of the primary early references of the subject [26]; but some of the foundations were laid in particular in work of Gromov on hyperbolic groups [37], Cannon on properties of the fundamental groups of compact hyperbolic manifolds [18], Gilman on groups with rational cross-sections [36]. The standard reference is certainly the book [26], but that is supplemented by some powerful results in [7, 34, 35], while Farb's article [31] gives a useful and readable overview of early development of the subject.

The definition of an automatic group was originally designed to identify properties of a group that were observed in the fundamental groups of compact hyperbolic 3-manifolds, and which facilitated computation with those groups. Such groups are finitely generated. When a group is automatic, its associated automatic structure allows the elements of the group to be represented as strings belonging to a particularly well structured set of strings, for which certain computations can be easily performed using finite state automata, as we shall see below.

Within this introductory section, we shall give some historical background, then define the notation and terminology that we shall need in the remainder of this chapter. Section 2 contains the definition of an automatic group, identifies the basic properties, and describes the most natural examples, and non-examples. Section 3 describes computation with automatic groups, how automatic structures may be computed, how they may, and have been, used. Section 4 describes how automaticity or biautomaticity of a group may be deduced from the geometry of a space on which the group has a good action. Section 5 describes the derivation of results proving automaticity or biautomaticity of groups in some well known families of group, which often used techniques or results described in Section 4. Finally Section 6 describes some problems that remain open.

1.1 Historical background

Alongside Thurston, it is natural to indentify Cannon, Epstein and Holt as the key figures in the early development of automatic groups. Much of the

information in this section comes from discussion with these three people [19, 30, 43], or can be found in the preface of the standard reference [26].

Cannon’s article [19] identifies the International Congress of Mathematicians in Helsinki in 1978 as a location at which key ideas that influenced the development of the concept of an automatic group were discussed.

In his plenary address, Thurston discussed the construction of geometric structures on a 3-manifold M , and the tessellation of its universal cover \tilde{M} by a structure dual to the Cayley graph of $\pi_1(M)$. Thurston’s geometrisation conjecture [72], subsequently proved by Perelman, claimed that every closed 3-manifold was geometrisable, that is, admitted a canonical decomposition into pieces each admitting one of eight types of geometric structure.

In his article [19], Cannon attributes to Thurston at that conference the conjecture that the growth series of a group G acting discretely, cocompactly and isometrically on a finite dimensional hyperbolic space \mathbb{H}_n should be a rational function. Cannon proved that conjecture in [18], where he identified features of \mathbb{H}_n within the Cayley graph $\text{Cay}(G, X)$ for G with respect to a finite generating set X . In particular, he proved that $\text{Cay}(G, X)$ admits finitely many types of “cones” on geodesics, and deduced from this the rationality of the growth function of G . Cannon also proved that the word and conjugacy problems for G could be solved using analogues of Dehn’s algorithms for those in hyperbolic surface groups. Gromov’s 1987 article [37] defined a combinatorial notion of hyperbolicity for a graph, and hence for a group (via its Cayley graph), and generalised Cannon’s results to groups satisfying this definition of hyperbolicity. There is a substantial body of material studying (Gromov) hyperbolic groups, in particular [2].

Thurston realised that the finiteness of the set of cone types in one of Cannon’s groups of hyperbolic isometries allowed the construction of a finite state automaton recognising the set of geodesic words within the group; rationality of the growth function is an immediate consequence of that set of words being the language of a finite state automaton. “Fellow travelling” properties of quasi-geodesic paths in \mathbb{H}_n that had been recognised by Cannon allowed the construction of further automata that recognised right multiplication in the group by a generator.

Now Thurston defined the concept of an automatic group. He called a group with finite generating set X *automatic* if it possessed a representative set of words L over X , such that one finite state automaton recognised the words in L , and other automata recognised pairs of words in L related in the group under right multiplication by the generators in X . Very early on, groups of this type were known as *regular groups* [43]. But this terminology conflicted with other uses of the term regular, and so was soon changed.

Initially, in particular in [26, 7], the study of the family of automatic groups was largely driven by the desire to find within it the groups of the geometrisable 3-manifolds, and hence to harness computational techniques that were provided by the association of automatic groups with regular languages. Epstein realised very early on that any automatic group must be finitely presented, while Thurston deduced that any such group had quadratic Dehn function and hence word problem soluble in quadratic time. Epstein and Holt in Warwick worked, together with the author of this chapter, to develop practical procedures to (attempt to) build automatic structures for finitely presented groups, and to compute within the groups using those structures.

1.2 Mathematical background and notation

All the groups that we consider will be finitely generated. If X is a finite generating set for a group G , then we write $G = \langle X \rangle$. In that case every element of G can be represented as a product (or string) of elements of X and their inverses. We denote by X^{-1} the set of symbols x^{-1} for which $x \in X$, and then by X^\pm the disjoint union of X and X^{-1} ; every non-identity element of G can now be described as a string of elements of X^\pm . The identity element, which we denote by 1, can be described as a product of length 0.

Given a finite set A , we define a *string* w over A to be a sequence $a_1 a_2 \cdots a_n$ with $a_i \in A$, and call n the *length* of w , denoted by $|w|$; we may alternatively use the term *word* over A rather than string. A subsequence $a_i a_{i+1} \cdots a_j$ of w is called a *substring* or *subword*. We write $w(i)$ for the *prefix* $a_1 \cdots a_i$ of w . We call the string or word of length 0 over A the *empty string* or *empty word* and denote that by ε . As is standard, we denote by A^+ the set of all strings over A of finite length > 0 and by A^* the union $A^+ \cup \{\varepsilon\}$. Given an ordering of the elements of A , we define the shortlex ordering on A^* as follows: for words $u = x_1 \dots x_r$ and $v = y_1 \dots y_s$, we define $v <_{\text{lex}} u$ if $|v| < |u|$, or if $|u| = |v|$ and for some i , $y_1 = x_1, \dots, y_{i-1} = x_{i-1}$ but $y_i < x_i$.

When X is a generating set for a group G , and $w \in (X^\pm)^*$, it is often convenient to abuse notation and use w to indicate not only that string over X^\pm but also the group element that the string represents; if $w, v \in X^\pm$, we write $w = v$ to denote that w, v are identical as strings, and $w =_G v$ to denote that w, v represent the same group element. If $g \in G$, we denote by $|g|$ the length of the shortest word over X^\pm that represents g . Suppose that $\text{Cay} = \text{Cay}(G, X)$ is the *Cayley graph* of G over X , that is the graph with vertex set G and, for each $g \in G$, $x \in X$, directed edges labelled x and x^{-1} connecting the ordered pairs of vertices (g, gx) and (gx, g) . Then for each $g \in G$, a path labelled by w joins the vertex g of Cay to the vertex gw ; we

shall represent that path as ${}_g w$.

When G is finitely generated by X , we define a *language* for G over X to be a subset of $(X^\pm)^*$ that contains at least one representative of each element of G , that is, that maps onto G under the map assigning each product over X to the element it represents.

For the free group F_n on a set X of n generators x_1, \dots, x_n , a language is provided by the set of all *freely reduced* words of length ≥ 0 over X^\pm , that is, the set of all words within which no subword $x_i x_i^{-1}$ or $x_i^{-1} x_i$ appears. For the free abelian group \mathbb{Z}^n on the same set of n generators, a language is provided by the set of all words of the form $x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_k}^{r_k}$, with $k \geq 0$, $i_1 < i_2 < \cdots < i_k$ and $r_i \in \mathbb{Z} \setminus \{0\}$. In each of these two examples the language provides a unique representative for each group element.

Each of the two languages just described is an example of a *regular language*, that is, it is the set $L(M)$ of strings accepted by a *finite state automaton* (fsa) M with alphabet $\{x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}\}$. Finite state automata provide standard models of bounded memory computation and are defined and studied in [46]. It is common to represent a finite state automaton M with alphabet A as a finite directed graph, with each directed edge labelled by one or more elements of A , one vertex identified as the *start*, and a subset of the vertices selected as *accepting*. A word w is then accepted by M if it labels at least one directed path from the start to an accepting vertex; if there is no such path, or if the end point (*target*) of every such path is a non-accepting vertex then w is not accepted. It is standard to call the vertices of M its *states*, the directed edges its *transitions* and the set of accepted words its *language* $L(M)$. In the cases where $n = 2$, the languages described above for the free and free abelian groups over $\{a, b\}$ are accepted by the two finite state automata shown in Figure 1; in each diagram, following convention, the start state is indicated by an arrow, and the accepting states are ringed. In each of the two examples, each of the five states shown in the diagram is accepting, but a further *failure state* is not shown, which constitutes a sixth state; the failure state is non-accepting, any transitions not shown in the diagram are assumed to be to that failure state, and all transitions from the failure state are to the failure state.

2 Automatic groups

2.1 Definition of an automatic group

Now suppose that G is a group with finite generating set X . For $k \in \mathbb{N}$, words w, v over X^\pm are said to *k-fellow travel* in G if for each $i \leq$

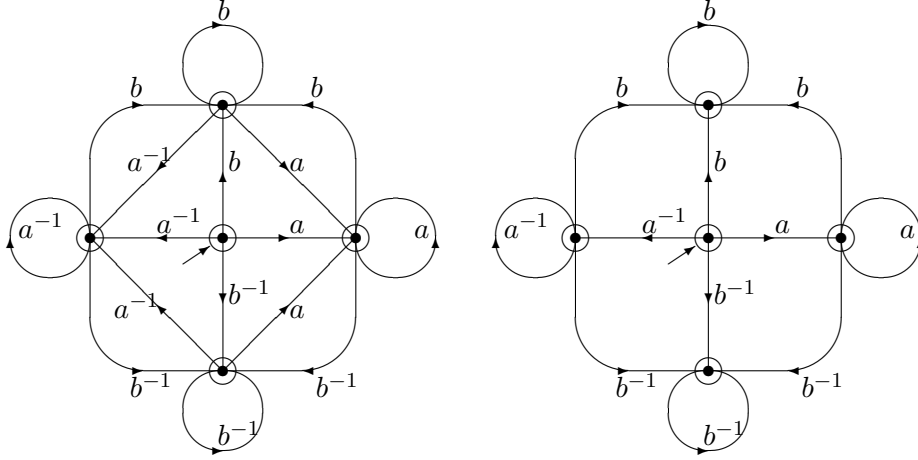


Figure 1: fsa M_1, M_2 giving the languages described in the text for F^2 and \mathbb{Z}^2

$\max\{|w|, |v|\}$ the distance between the vertices $w(i)$ and $v(i)$ of $\text{Cay} = \text{Cay}(G, X)$ (using the graph metric) is at most k . Equivalently, we say that the paths ${}_1w$ and ${}_1v$ of Cay k -fellow travel. A group G with finite generating set X is defined to be *automatic* over X if

A1 there is a language L for G over X that is regular,

A2 there is an integer k such that, for each $y \in X \cup \{1\}$, and for any $w, v \in L$ with $wy =_G v$, the paths ${}_1w, {}_1v$ k -fellow travel in Cay .

We call L the *language*, the fsa accepting L the *word acceptor* and k the *fellow traveller constant* of an *automatic structure* for G .

The fsa M_1 illustrated in Figure 1 is the word acceptor of an automatic structure with fellow traveller constant 1 for F_2 over $\{a, b\}$; each element of the group has a unique representative in the language, and given two words $w, v \in L(M_1)$ and $y \in \{a^{\pm 1}, b^{\pm 1}\}$ with $wy =_{F_2} v$, one of the words is a maximal prefix of the other, and so the words 1-fellow travel in G .

Similarly, the fsa M_2 of Figure 1 is the word acceptor of an automatic structure with fellow traveller constant 2 for \mathbb{Z}^2 over $\{a, b\}$. Again each element of the group has a unique representative in the language, and given two words $w, v \in L(M_2)$ and $y \in \{a^{\pm 1}, b^{\pm 1}\}$ with $wy =_{\mathbb{Z}^2} v$, corresponding vertices on the paths ${}_1w$ and ${}_1v$ in $\text{Cay}(\mathbb{Z}^2, \{a, b\})$ are joined in the graph by a path of length 1 or 2. The language $L(M_2)$ is the set of all SHORTLEX minimal geodesic representatives of group elements; we call this a *shortlex automatic structure* for \mathbb{Z}^2 . Note that we can define a similar shortlex automatic structure for \mathbb{Z}^n .

In the definition of automaticity given in [26] the condition A2 given above is replaced by the following condition:

A2' For each $y \in X \cup \{\{1\}\}$, the set of pairs (w, v) for which $w, v \in L$ and $wy =_G v$ is a regular language when viewed as a set of strings over the alphabet of pairs $\{(a, b) : a, b \in X^\pm \cup \{\$\}\}$; the character $\$$ is a *padding symbol* used to deal with the situation where $|w| \neq |v|$, in which case the shorter of the two words is padded with $\$$ s at its end.

The automata recognising the regular languages just described are known as the *multiplier automata* of the automatic structure, usually denoted by M_y , for each choice of y .

In the presence of A1 the conditions A2 and A2' are equivalent. This is a consequence of the fact that the k -fellow travelling of a pair of words w, v can be tracked by an automaton whose state set \mathcal{D} corresponds to a set of words of length at most k ; a pair of words (w, v) is accepted by that automaton so long as all the products $w'^{-1}v'$ associated with prefixes $w' := w(i), v' := v(i)$ of w, v are represented by words in \mathcal{D} . We call such an automaton a *word difference machine*, and the associated set \mathcal{D} its corresponding set of *word differences*.

Where G is automatic over its finite generating set X , with automatic structure L, k , then G is said to be *biautomatic* (and (L, k) to be a *biautomatic structure* for G) if the additional condition A3 is satisfied:

A3 for each $y \in X$, and for any $w, v \in L$ with $yw =_G v$, the paths $_y w, {}_1 v$ k -fellow travel in $\text{Cay}(G, X)$.

This further fellow traveller condition can be expressed in terms of **fsa** that recognise left multiplication, usually denoted by $_y M$, for $y \in X$. It is an open question whether all automatic groups are biautomatic.

The concept of automaticity can be generalised to one of *asynchronous automaticity* by replacing the fellow traveller condition by an *asynchronous fellow travel condition*; for two words w, v to asynchronously fellow travel within a group G it is the distance between vertices $w(j_i)$ and $v(k_i)$ that must be bounded, where, for some $m \geq \max(|w|, |v|)$, the sequences (j_0, j_1, \dots, j_m) and (k_0, \dots, k_m) are both increasing sequences of integers, with $j_0 = k_0 = 0$, $j_m = |w|$, $k_m = |v|$, and for $0 \leq l < m$, $j_{l+1} - j_l$ and $k_{l+1} - k_l$ are in $\{0, 1\}$. Asynchronous automaticity is certainly a more general concept than automaticity, and it is satisfied by examples such as the Baumslag–Solitar groups which are certainly not automatic.

It is fairly standard to call a language L for a group G that satisfies the condition A2 (but not necessarily A1) a *combing* for G , and a language that satisfies both A2 and A3 a *bicombing* for G ; however some authors use these terms differently, e.g. impose additional (geometric) conditions on L . Again, the fellow travelling condition can be replaced by an asynchronous one, in order to define asynchronous combings and bicomblings. The basic properties of combable groups are studied in [12], where it is proved that non-automatic combable groups exist (answering a question posed in [26]), as well as combable groups that are not bicomable.

Given an automatic (or biautomatic) structure (L, k) for a group G , it is straightforward (using well known properties of regular languages, such as the “Pumping lemma” [46]) to modify the structure and achieve a new automatic structure with particular properties. For instance we can achieve a structure in which every element of G has a unique representative (a structure *with uniqueness*) a *prefix closed* structure in which the language contains every prefix of every one of its elements, a *quasigeodesic* structure in which every element is represented by a (λ, ϵ) -quasigeodesic. We note that a word w representing an element g of a group G is called a (λ, ϵ) -*quasigeodesic* if every subword w' of w has length at most $\lambda|g'| + \epsilon$, where g' is the element represented by w' . Note that it is not clear that all combinations of properties can be achieved within the language of a single automatic structure. In particular it is an open question [26] whether, given an automatic structure for a group G , an automatic structure can be derived for G that is both prefix closed and has uniqueness.

Note that the definitions of automaticity and biautomaticity are independent of choice of generating set; that is if G has an automatic structure over a finite generating set X , then it has one over any other finite generating set Y .

2.2 Basic properties of automatic groups

Some properties of automatic groups can be deduced very easily from basic properties of regular languages, which imply certain constraints on their Cayley graphs. In particular any automatic group is finitely presented with soluble word problem, and quadratic Dehn function, while any biautomatic group has soluble conjugacy problem. We recall that the word problem is soluble in G if an algorithm exists that can decide whether or not any input word represents the identity, and the conjugacy problem is soluble if an algorithm exists that can decide whether or not two input words represent elements that are conjugate within the group; it is an open question whether the conjugacy problem is soluble for automatic groups. It also is an open

question whether the isomorphism problem is soluble for automatic groups, that is, whether an algorithm that was given as input automatic structures for a pair of groups G, H could decide whether or not G and H were isomorphic. It is conjectured in [26] that this problem is insoluble. Note that it is soluble for hyperbolic groups [24, 70].

In order to explain these statements in more detail, we use the language of van Kampen diagrams. Informally (essentially, following [53]), given a group G with presentation $\langle X \mid R \rangle$ and a word w over X that represents the identity of G , we define a van Kampen diagram Δ_w for w to be a finite, connected, directed, planar graph, with a selected *basepoint*, whose directed edges are labelled by elements of X , in such a way that the boundary of every face of the graph (known as a *cell*) is labelled (from some starting point, in some orientation) by a word from R , while the boundary of the graph is labelled (from the basepoint) by w . As a directed, edge labelled graph, Δ_w maps (not necessarily injectively) into the Cayley graph $\text{Cay}(G, X)$. The *area of the diagram* $\text{Area}(\Delta_w)$ is defined to be the number of cells it contains; of course its value is dependent on the set R , and would change if R were changed.

We define the *area of the word* w to be the minimum of the areas of all van Kampen diagrams that represent w . And we define the *Dehn function* (or *isoperimetric function*) for G , $f : \mathbb{N} \rightarrow \mathbb{N}$, to be the function for which $f(n)$ is the maximum area of all words w of length n over X^\pm that represent the identity of G . Although the precise form of the Dehn function depends on the chosen presentation for G , it can be shown that two Dehn functions corresponding to different presentatives are related by a natural notion of equivalence, and in particular if one is polynomially bounded, then both are, by polynomials of the same degree.

Proposition 2.1. *Every automatic group is finitely presented, with a quadratic upper bound on the Dehn function, and hence soluble word problem.*

We sketch the proof, which is that of [45, Theorem 5.2.13].

Proof. We suppose that L, k are the language and fellow traveller constant of an automatic structure over a generating set X ; we may assume that L consists of quasigeodesics. Suppose that $w = a_1 \cdots a_n$ is a word of length n representing the identity. Now we define words w_0, \dots, w_n as follows. We define $w_0 = w_n$ to be a representative in L of 1, and for each $i = 1, \dots, n-1$ we choose w_i to be a representative in L of the prefix of w of length i ; since L is quasigeodesic, we can choose w_i of length at most $|w_0| + Ci$, for some constant C of the automatic structure. We start with a disk within the plane whose boundary is labelled by w , and divide it into cells to form a

van Kampen diagram Δ_w with boundary w as follows. First, a loop labelled by w_0 connects the basepoint to itself, while for each i a path labelled w_i connects the basepoint to the point on the boundary distance i along w , and none of these paths cross each other. Then, since the paths ${}_1w_{i-1}$ and ${}_1w_i$ in $\text{Cay}(G, X)$ fellow travel at distance at most k , we can construct paths of length at most k that connect corresponding vertices on the paths within the disk labelled by those two words, and hence divide the region between the two paths into cells each of length at most $2k + 2$. In this way we divide the interior of the diagram into a number of cells labelled by words of length at most $2k + 2$, together with two cells labelled by the word $w_0 = w_n$, as illustrated in Figure 2.

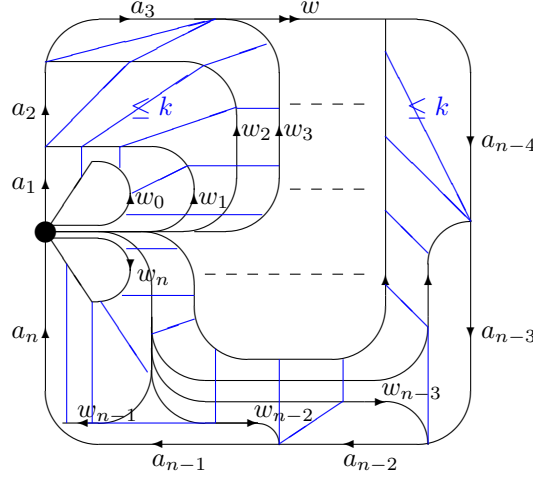


Figure 2: Van Kampen diagram for a representative of the identity in an automatic group

Using the bounds on $|w_i|$, we see that the total number of cells is bounded by a quadratic function of n . We now define R to be the set of all words of length up to $2k + 2$ that represent the identity, together with the word w_0 . Then $\langle X \mid R \rangle$ is a finite presentation for G , and, relative to R , Δ_w has quadratic area. \square

A similar argument proves an exponential upper bound on the Dehn function for any asynchronously automatic group; it is an open question [26] whether a polynomial time solution to the word problem must exist.

The most straightforward way to prove a group non-automatic is probably to show that it has a Dehn function that is above quadratic. This argument proves easily the non-automaticity of the Baumslag–Solitar groups $\langle a, b \mid ba^p b^{-1} = a^q \text{ for which } p, q > 0 \text{ and } p \neq q \rangle$, since they have exponential Dehn

function; in fact they provide examples of non-automatic groups that are asynchronously automatic.

But there are many groups with quadratic Dehn functions that are known by other methods not to be automatic.

The non-automaticity of the groups $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ is proved in [26]. The group $\mathrm{SL}_2(\mathbb{Z})$ is well known to be virtually free, and hence hyperbolic with linear Dehn function. The group $\mathrm{SL}_3(\mathbb{Z})$ has exponential Dehn function and so is certainly non-automatic. However, the existence of a quadratic Dehn function for $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 5$ was proved in [77] in 2013 (and had been conjectured by Thurston, in fact for $n \geq 4$). In order to prove non-automaticity of the group for all $n \geq 3$, Epstein and Thurston derived higher dimensional isoperimetric inequalities that would have to hold in any combable group of isometries acting properly discontinuously with compact quotient on a k -connected Riemannian manifold [26, Theorem 10.3.5]. The non-automaticity of $\mathrm{SL}_n(\mathbb{Z})$ now follows by the construction of a proper discontinuous cocompact action on a suitable contractible manifold, and the demonstration that a higher dimensional isoperimetric inequality fails; hence $\mathrm{SL}_n(\mathbb{Z})$ is proved to be non-combable and so non-automatic.

Van Kampen diagrams can also be used to prove solubility of the conjugacy problem in any biautomatic group, by demonstrating the existence of a conjugator of bounded length. The proof below, valid for any bicomvable group, is taken from [71]; an earlier result of [35] constructs an automaton out of the biautomatic structure to solve the problem.

Proposition 2.2. *Given a biautomatic group G , any two words u, v representing conjugate elements are conjugate by an element of length at most $a^{|u|+|v|}$, for some constant a (depending only on the biautomatic structure). Hence any biautomatic group has soluble conjugacy problem.*

Proof. We choose a biautomatic structure (L, k) over a finite generating set X , and suppose that the words u, v over X^\pm represent conjugate elements of G . Let $N := |X^\pm|^{k(|u|+|v|)}$. We find a conjugator of length at most N , and so $a = |X^\pm|^k$.

For suppose that an element $g \in G$ conjugates u to v , that is that $gu =_G vg$, and that $w, w' \in L$ represent the elements g and ug , respectively. We consider the paths ${}_1w, {}_1w'$ and ${}_uw$ within the Cayley graph $\mathrm{Cay}(G, X)$, and see that the biautomaticity of G ensures that ${}_1w$ and ${}_1w'$ fellow travel at distance at most $|u|k$, and that ${}_1w'$ and ${}_uw$ fellow travel at distance at most $|v|k$. We deduce that we can construct a van Kampen diagram with boundary labelled by $wuw^{-1}v^{-1}$ in which chords of length at most $(|u| + |v|)k$ join boundary vertices in corresponding positions on the two

boundary subwords labelled by w , as shown on the left hand side of Figure 3. Where $|w| = n$, let d_1, d_2, \dots, d_{n-1} be the words that label those chords.

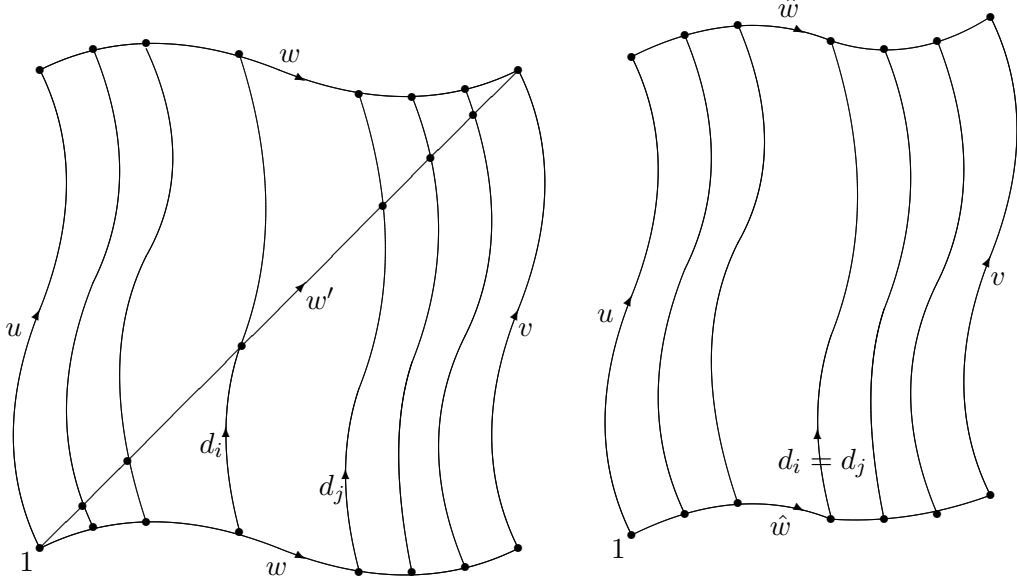


Figure 3: Finding a conjugator of bounded length

Now if $n > |X^\pm|^{(|u|+|v|)k}$, then for some i, j we have $d_i = d_j$. In that case, where \hat{w} is the word formed from w by deleting its middle section of length $j - i$, from its $(i + 1)$ -th to its j -th letter, we can form the van Kampen diagram with boundary word $\hat{w}u\hat{w}^{-1}v^{-1}$ shown on the right hand side of Figure 3 by deleting the central part of the diagram we already constructed for $wuw^{-1}v^{-1}$.

□

Various combinations of automatic groups are known to be automatic [26, 7]: these include free products, direct products, certain amalgamated products and HNN extensions of automatic groups, as well as subgroups of finite index in automatic groups, groups with automatic groups as subgroups of finite index, quotients of automatic groups by finite normal subgroups. Some, but not all, of these closure properties also hold for biautomatic groups. It is an open question whether direct factors of automatic groups must be automatic (but the analogous result is proved for biautomatic groups [61]). It is also open [26] whether a group with a biautomatic group as a subgroup of finite index must be biautomatic.

2.3 Basic examples and non-examples

2.3.1 Virtually abelian groups, soluble groups

We already described shortlex automatic structures for the free abelian group \mathbb{Z}^n . In fact \mathbb{Z}^n is also biautomatic, but with a different (less straightforward) language, and indeed so is every virtually abelian group. However it was already proved in [26] that an automatic nilpotent group must be virtually abelian; the proof uses the fact that a regular language with polynomial growth cannot satisfy a (synchronous) fellow traveller property. It was conjectured by Thurston that the same result must hold for an automatic soluble group. That conjecture remains open, but it was proved for automatic polycyclic groups in [39], using an embedding of a finite index subgroup of a polycyclic group of exponential growth as a lattice in an appropriate Lie group, where [26, Theorem 10.3.5] about higher dimensional isoperimetric functions could be applied, which had previously been used to prove the non-automaticity of $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$. Much more recently it was proved in [69] that biautomatic soluble groups must be virtually abelian.

2.3.2 Hyperbolic groups

Maybe the most natural examples of non-abelian automatic groups are provided by the large family of *word hyperbolic* groups, which contains all finitely generated free groups as well as the fundamental groups of all compact hyperbolic manifolds.

A group G with finite generating set X is said to be *word hyperbolic* if its Cayley graph $\mathrm{Cay}(G, X)$ is a δ -hyperbolic metric space, for some $\delta \geq 0$; a geodesic metric space (\mathcal{X}, d) is δ -hyperbolic if for any triangle in \mathcal{X} with geodesic sides $\gamma_1, \gamma_2, \gamma_3$ and for any vertex p on the side γ_1 there is a vertex q on the union $\gamma_2 \cup \gamma_3$ of the other two sides for which $d(p, q) < \delta$ (we say that triangles in \mathcal{X} are δ -*slim*). The property of being word hyperbolic is independent of the choice of a finite generating set for G , although the value of δ is not. The fundamental groups of compact hyperbolic manifolds give examples, as do finitely generated free groups (which are 0-hyperbolic with respect to free generating sets).

We note that there are many equivalent definitions of hyperbolicity for metric spaces (and hence for finitely generated groups), which are explained in [2]. In particular there is a characterisation in terms of *thin* rather than *slim triangles* (and a linear relationship between the associated parameters “ δ ”).

It is proved in [26] that a word hyperbolic group G is automatic over any

generating set X , with an automatic structure whose language consists of all geodesic words over the selected generating set. The regularity of that set of geodesic words is equivalent to the fact that the Cayley graph $\text{Cay} = \text{Cay}(G, X)$ contains finitely many cone types. For $g \in G$, represented by a geodesic word w , we define the *cone* $C(g)$ (or $C(w)$) on the vertex g of Cay to be the set of (geodesic) paths γ within Cay starting at g for which the concatenation $\eta\gamma$ of a geodesic path η from 1 to g with γ is also geodesic. The *cone type* $[C(g)]$ or $[C(w)]$ of the cone is defined to be the set of words that label the paths within it. Now for $y \in X \cup X^{-1}$, if wy is also geodesic then for any word v ,

$$v \in [C(wy)] \iff yv \in [C(w)].$$

It follows that we can recognise the set of geodesic words over X^\pm with an fsa whose states correspond to the cone types, with a transition from $[C(w)]$ to $[C(wy)]$ on y whenever wy is geodesic, but otherwise to a single *failure state* (i.e. a non-accepting sink state). We can illustrate this construction in the free abelian group \mathbb{Z}^2 with generating set $\{a, b\}$, where there are nine cone types $[C(w)]$, defined by the nine geodesic words $\varepsilon, a, b, a^{-1}, b^{-1}, ab, ab^{-1}, a^{-1}b, a^{-1}b^{-1}$, and consisting of the nine possible sets of geodesic words in which each generator appears either only with positive exponent, or only with negative exponent, or not at all. The fsa is illustrated in Figure 4. This automaton is *not* part of an automatic structure for \mathbb{Z}^2 ; it cannot be since, for example, the vertices distance i from the origin on the geodesic words $a^i b^i$ and $b^i a^i$ are distance $2i$ apart within the Cayley graph, and hence this language does not satisfy a fellow travelling property.

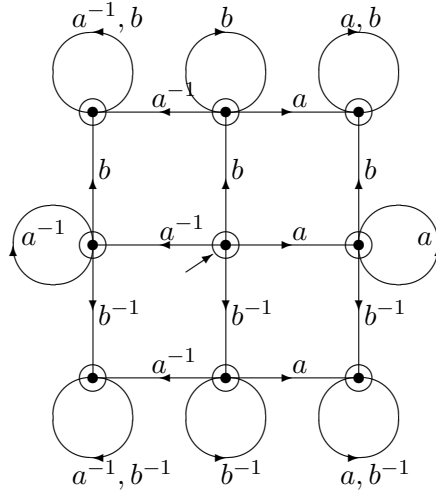


Figure 4: fsa recognising geodesics in \mathbb{Z}^2

Given the finiteness of the set of cone types in a word hyperbolic group, biautomaticity of any word hyperbolic group now follows once it is observed that

the fellow travelling of two geodesic words with common (or adjacent) start and end vertices can be derived from the slimness of triangles. In fact, it is proved by Papasoglu [67] that this fellow traveller condition characterises word hyperbolic groups, and hence so does the existence of a (bi)automatic structure that consists of all geodesic words. A procedure to test for hyperbolicity that is based on this result is described in [75]. Starting with a shortlex automatic structure (L, k) for a group G over X , the procedure attempts to construct an automatic structure $(\widehat{L}, \widehat{k})$ with $\widehat{L} \supset L$ and $\widehat{k} \geq k$, and such that \widehat{L} contains all geodesic words over X^\pm . It will terminate with such a structure precisely when G is hyperbolic. An improved procedure, based on the same result was developed by Holt and Epstein [27] and implemented in KBMAG.

The fundamental groups of finite volume hyperbolic manifolds (geometrically finite hyperbolic groups) were proved biautomatic by Epstein [26], with a further biautomatic structure subsequently described by Lang [55].

Geometrically finite hyperbolic groups were the motivating examples for Bowditch's definition [9] of a group hyperbolic relative to a collection of subgroups; a geometrically finite hyperbolic group is hyperbolic relative to a collection of abelian groups. The major part of the definition of relative hyperbolicity is the requirement that the Cayley graph of a group hyperbolic relative to a collection \mathcal{H} of subgroups becomes hyperbolic after the contraction of edges within left cosets of subgroups in \mathcal{H} . However weaker and stronger versions of the definition exist depending on whether or not a condition of *bounded coset penetration* is required to hold. Under the stronger definition (studied in [66]) it is proved, in particular in [5], that groups hyperbolic relative to shortlex biautomatic subgroups are themselves shortlex biautomatic. The shortlex biautomaticity of geometrically finite hyperbolic groups is a consequence of this result.

A further generalisation of hyperbolic groups is provided by semihyperbolic groups, which were introduced by Bridson and Alonso in [3]; the class contains all biautomatic groups (hence all hyperbolic groups) and all CAT(0) groups (see Section 4). A group G with finite generating set X is defined to be *weakly semihyperbolic* if $\text{Cay}(G, X)$ admits a bounded quasi-geodesic bicombing (with a unique combing path $s_{g_1, g_2}(t)$ identified between any pair g_1, g_2 of vertices of the graph), and *semihyperbolic* if it has such a bicombing that is equivariant under the action of G (so that $g.s_{g_1, g_2}(g) = s_{gg_1, gg_2}(t)$). This class of groups satisfies many closure properties, and all groups within it are finitely presented, with soluble word and conjugacy problems.

2.3.3 Fundamental groups of compact 3-manifolds

It is proved in [26] that the fundamental groups of compact 3-manifolds based on six of Thurston's eight model geometries for compact 3-manifolds [73] admit automatic structures. But it is also proved that the fundamental groups of closed manifolds based on the Nil and Sol geometries (which are non-abelian, nilpotent and soluble, respectively) cannot even be asynchronously automatic [26, 10].

However, using combination theorems for automatic groups, it can be proved (as in [26, Theorem 12.4.7], but our wording is slightly different) that an orientable, connected, compact 3-manifold with incompressible toral boundary whose prime factors have JSJ decompositions containing only hyperbolic pieces has automatic fundamental group. It was proved in [13, Theorem B] that the fundamental group of a manifold as above in which manifolds based on Nil and Sol are allowed within the JSJ decomposition, while not automatic, still admits an asynchronous combing based on an indexed language [1].

3 Computing with automatic groups

3.1 Building automatic structures

The original motivation for the definition of automatic groups was computational, and so it was important from the beginning of the subject to be able to construct automatic structures, that is, given a presentation for a group G , to have a mechanism for building the word acceptor and multiplier automata of an associated automatic structure. Software to build these automata was developed at the University of Warwick, and the procedure used is described in [29]. The original programs were subsequently rewritten by Holt, and released within his KBMAG package [54], now available within both GAP and Magma computational systems [33, 59].

The basic procedure is the same in both versions (the ideas are due to Holt) and we describe it briefly now, but refer the reader to [29] or [42] for more details.

A presentation for a group G over a finite generating set X is input, together with an ordering of the set X^\pm . The procedure attempts to prove G to be shortlex automatic over X (with the given ordering) by first constructing a set of automata consisting of W and M_y for $y \in X^\pm \cup \{\varepsilon\}$, and then attempting to verify that those automata are indeed the automata of a

shortlex automatic structure. If verification tests fail, some looping is possible within the procedure, and indeed that looping could continue indefinitely (or at least until the computer runs out of resources). If all verification tests pass, then the procedure will have verified the shortlex automaticity of G by construction and checking of a shortlex automatic structure.

So the procedure may succeed in proving shortlex automaticity of G . But if it fails, it has certainly not proved that G is not automatic, or even that G is not shortlex automatic, but rather it suggests that G is unlikely to be shortlex automatic over the given generating set X , with the given ordering of the elements of X . We note that the question of automaticity for a finitely presented group is undecidable in general; this follows from the undecidability of questions such as triviality for a group. We note too that it is an open question [26] whether every automatic group must be shortlex automatic with respect to some ordered generating set.

The first step of the procedure to prove shortlex automaticity is the construction of a rewrite system \mathcal{R} from the group presentation that is compatible with the shortlex order. By definition, \mathcal{R} is a set of substitution rules $\rho : u \rightarrow v$, for $u, v \in (X^\pm)^*$, and with $v <_{\text{slex}} u$; in order that \mathcal{R} encodes the presentation we require that every relator from the group presentation is a cyclic conjugate of the product uv^{-1} or its inverse for at least one such rule.

The next step is to run the Knuth–Bendix procedure for a while on \mathcal{R} . The Knuth–Bendix procedure (described in [45]) is a general procedure that, given as input a rewrite system \mathcal{R} for strings compatible with a partial order, modifies it by adding rules that are consequences of existing rules and deleting rules that have become redundant, in order to produce a new rewrite system. The procedure attempts to build a finite *complete* system, for which any input word w can be rewritten after a finite number of steps to a unique irreducible word w' (where irreducible means that w' cannot be rewritten further). However with this goal the procedure may never terminate; all that is guaranteed is that after bounded time the modified system must contain enough rules to reduce any word up to some bounded length to an irreducible.

In fact the procedure to construct a shortlex automatic structure for G does not need the Knuth–Bendix procedure to terminate on the input rewrite system \mathcal{R} . Instead, while the Knuth–Bendix procedure is running it accumulates the set \mathcal{D} of *word differences* $u(i)^{-1}v(i)$ and their inverses (reduced according to the current modification of \mathcal{R}) that correspond to prefixes of the rules $u \rightarrow v$ in the system. Where $u = u_1 \cdots u_m$, and $v = v_1 \cdots v_{m'}$, a transition is added from each word difference $u(i)^{-1}v(i)$ to $u(i+1)^{-1}v(i+1)$, creating a *word difference machine* that can recognise fellow travelling with respect to \mathcal{D} .

The Knuth–Bendix procedure is paused when it seems that the set \mathcal{D} and the associated automaton have stabilised. And then a candidate word acceptor WA is constructed, designed to reject a word u if a string v exists with $v <_{\text{slex}} u$ for which (u, v) fellow travels according to \mathcal{D} while also the word difference $u^{-1}v$ reduces, according to the current rewrite system, to the empty word.

Similarly, multiplier automata are constructed for each $y \in X^\pm \cup \{\varepsilon\}$, using a direct product construction on automata to recognise pairs of words u, v for which $u, v \in L(W)$, (u, v) fellow travels according to \mathcal{D} , while also the word difference $u^{-1}v$ reduces, according to the current rewrite system, to y .

Now a series of elementary tests is applied to the candidate automata. If some of these tests fail, then \mathcal{D} has been proved to be inadequate, and the Knuth–Bendix procedure is restarted. If and when those tests are passed, further tests known as *axiom checking* are applied, and a positive result for these tests proves the automata to provide a shortlex automatic structure for G . If the axiom checks fail then the procedure is abandoned.

3.2 Calculation using the automatic structure

Once an automatic structure has been constructed for a group G , much can be computed using the automata of that structure. Various of these functions are available within the KBMAG package [54].

It is straightforward to enumerate the language of a finite state automaton. Hence we can enumerate a set of representative words for an automatic group, with unique representation if necessary (recall that once an automatic structure has been derived, a structure with unique representation can be derived from that).

For any regular language L the generating function $\sum_{n=0}^{\infty} s_L(n)x^n$, where $s_L(n)$ denotes the number of words of length n in L , is a rational function, and can be computed from an automaton recognising L . Hence the growth series of an automatic group is computable, given a geodesic automatic structure.

Reduction of an input word to the “normal form” defined by the language L of the automatic structure for G can be performed using a combination of the word acceptor and multiplier automata, or alternatively using the word difference machine.

Finiteness or otherwise of an automatic group is immediately recognisable from a word acceptor for an automatic structure; the language is infinite

precisely when the automaton admits loops. In this way, the Heineken group $G = \langle x, y, z \mid [x, [x, y]] = z, [y, [y, z]] = x, [z, [z, x]] = y \rangle$ was proved infinite, by Holt using KBMAG; computation with the automatic structure subsequently revealed the group to be hyperbolic. Previously that group had been proposed as a possible example of a finite group with a balanced presentation. Similarly, a second proof of the infiniteness of the Fibonacci group $F(2, 9)$ was provided by the construction of an automatic structure for it [41].

Tests for hyperbolicity [75, 27] that make use of automatic structures for G together with Papasoglu’s characterisation of hyperbolic groups have already been described in Section 2.3.2. The second of those is implemented in KBMAG, as is an algorithm [27] estimating the thinness constant (related to, but not equal to, the slimness constant) for geodesic triangles in the Cayley graph of a word hyperbolic group.

Quadratic and linear time solutions to the conjugacy problem in a hyperbolic group are described in [15] and [28, 17]. A practical cubic time solution that restricts to infinite order elements is due to Marshall [58], using some ideas from Swenson, and has been implemented in the GAP system.

4 Group actions and negative curvature

One of the basic principles of geometric group theory is generally referred to as the Milnor–Švarc lemma:

If a group G has a “nice” (properly discontinuous and cocompact) discrete, isometric action on a metric space \mathcal{X} then its Cayley graph is quasi-isometric to \mathcal{X} . In particular a group with such an action on a δ -hyperbolic space is word hyperbolic.

A variety of results derive automaticity or biautomaticity of a group from its “nice” actions on spaces in which some kind of non-positive curvature can be found.

Theorem 4.1 (Gersten, Short, 1990, 1991 [34, 35]). *A group acting discretely and fixed point freely on a piecewise Euclidean 2-complex of type $A_1 \times A_1$, A_2 , B_2 or G_2 (corresponding to tessellations of the Euclidean plane by squares, equilateral triangles, or triangles with angles $(\pi/2, \pi/4, \pi/4)$ or $(\pi/2, \pi/3, \pi/6)$) is biautomatic.*

As a consequence of the above results, and within the same two articles, Gersten and Short deduce that groups satisfying any of the small cancellation conditions $C(7)$ or else $T(p)$ and $T(q)$ with $(p, q) \in \{(3, 7), (4, 5), (5, 4)\}$

(defined in [57]) are hyperbolic, and hence in particular biautomatic, and then that groups satisfying the small cancellation conditions $C(6)$, or $C(4)$ and $T(4)$, or $C(3)$ and $T(6)$ are biautomatic.

A geodesic metric space \mathcal{X} is defined to be $CAT(0)$ if for any geodesic triangle in the space, and for any two points p, q on the sides of that triangle, the distance between p and q in \mathcal{X} is no more than the distance between the points in corresponding positions on the sides of a geodesic triangle with the same side lengths in the Euclidean plane, as illustrated in Figure 5. A complete $CAT(0)$ space is often called a *Hadamard space*. A group is called $CAT(0)$ if it acts properly and cocompactly on a $CAT(0)$ space.

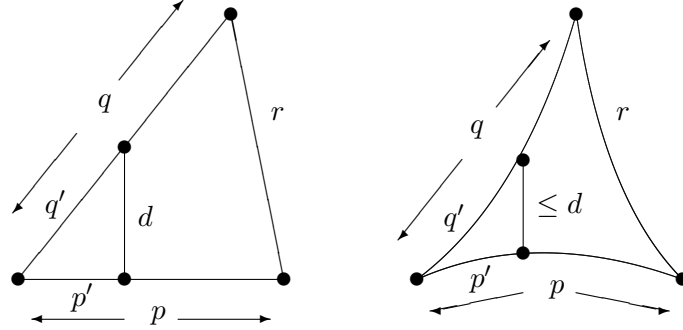


Figure 5: Comparable triangles in Euclidean and $CAT(0)$ spaces

The $CAT(-1)$ condition is defined similarly with respect to the hyperbolic plane; any $CAT(-1)$ space is δ -hyperbolic, for some δ , and hence $CAT(-1)$ groups are word hyperbolic.

A (not necessarily geodesic) metric space (X, d) is said to have non-positive curvature (or curvature ≤ 0) if every point of X is contained in a $CAT(0)$ neighbourhood. By the Cartan-Hadamard theorem [14] the universal cover of a complete connected space of non-positive curvature is $CAT(0)$.

Niblo and Reeves studied in particular groups acting on $CAT(0)$ cube complexes:

Theorem 4.2 (Niblo, Reeves 1998 [64]). *A group acting faithfully, properly discontinuously and cocompactly on a simply connected and non-positively curved cube complex is biautomatic.*

A cube complex is defined to be a metric polyhedral complex in which each cell is isometric to the Euclidean cube with side lengths 1, where the gluing maps are isometries. Such a complex is non-positively curved provided that it contains at most one edge joining any two vertices, and no triangles of

edges, and (by a result of Gromov [37]) is CAT(0) if non-positively curved and simply connected.

Actions of Coxeter groups on CAT(0) cube complexes are constructed in [65], but are not necessarily cocompact. However in some cases it follows from those or related constructions that the Coxeter groups are biautomatic (see Section 5.2).

There are many open problems relating to CAT(0) groups (see for example [32]). The question of whether every CAT(0) group must be biautomatic was recently resolved in the negative by Leary and Minasyan [56], who constructed an example of a 3-dimensional CAT(0) group which could admit no biautomatic subgroup of finite index. It is still unknown whether non-automatic CAT(0) groups can exist.

However a restricted class of CAT(0) groups is provided by groups that act geometrically on CAT(0) spaces with isolated flats. A k -flat in a CAT(0) space is an isometrically embedded copy of Euclidean space \mathbb{R}^k . This family contains a number of interesting examples, including geometrically finite Kleinian groups, the fundamental groups of various compact manifolds, and limit groups, arising from the solutions of equations over free groups. Groups of this type are studied in [48], where more details (of definition and examples) can be found. Theorem 1.2.2 of that article establishes a number of properties of such groups, including their biautomaticity.

A form of non-positive curvature in simplicial complexes is defined in [52]: a flag simplicial complex \mathcal{X} is called k -systolic if connected, simply connected and locally k -large (no minimal ℓ -cycle with $3 < \ell < k$ in the link of a vertex). A group is called k -systolic if it acts simplicially, properly discontinuously and cocompactly on a k -systolic simplicial complex, and is called systolic if 6-systolic.

Theorem 4.3 (Januszkiewicz, Swiatkowski, 2006 [52]). *7-systolic groups are hyperbolic, 6-systolic groups are biautomatic.*

This result is used to prove biautomaticity of a large class of Artin groups [49], as detailed in Section 5.1.

A *Helly graph* is a graph in which every family of pairwise intersecting balls has a non-empty intersection. A group is called Helly if it acts properly and cocompactly by graph automorphisms on a Helly graph; word hyperbolic groups, CAT(0) cubical groups and C(4)-T(4) small cancellation groups are all examples. It is proved in [22] that all Helly groups are biautomatic. This result is used to prove biautomaticity of another large class of Artin groups [50], as detailed in Section 5.1.

5 Some automatic and biautomatic families

Over a period of more than 30 years, automatic and biautomatic structures were found for various families of groups, including braid groups, many Artin groups, mapping class groups, and Coxeter groups. But some questions remain open for these families.

5.1 Braid groups, Artin groups and Mapping Class groups

Automatic structures for the braid group B_n on n strands and also for the (closely related) mapping class group of the $(n + 1)$ -punctured sphere were constructed by Thurston and are described in [26]; one of the structures described for the braid groups is symmetric, proving the braid groups to be biautomatic. The automaticity (but not necessarily biautomaticity) of the mapping class group of the $n + 1$ -punctured sphere then follows from the fact that it contains the quotient of the braid group B_n by its centre as a subgroup of index $n + 1$.

The braid group on $n + 1$ strands is isomorphic to the Artin group of finite type A_n . We recall that an Artin group is a group defined by a presentation of the form

$$\langle x_1, x_2, \dots, x_n \mid \overbrace{x_i x_j x_i \cdots}^{m_{ij}} = \overbrace{x_j x_i x_j \cdots}^{m_{ij}}, i \neq j \in \{1, 2, \dots, n\} \rangle,$$

relating to a symmetric, integer *Coxeter matrix* (m_{ij}) , or equivalently a *Coxeter diagram* Γ on n vertices, whose edge $\{i, j\}$ is labelled m_{ij} , and is naturally associated with a Coxeter group by adding relations $x_i^2 = 1$ for each i . The Artin group has finite type if the associated Coxeter group is finite (and hence Γ is a disjoint union of diagrams from the well-known list of spherical Coxeter diagrams).

In [23], Charney used results of Deligne to extend Thurston's construction for the braid groups to all finite type Artin groups. Charney's construction provided biautomatic structures for all finite type Artin groups; these biautomatic structures were geodesic over the "Garside" generating sets, but not over the standard generators x_i . Biautomatic structures for all Garside groups (of which finite type Artin groups are examples) were described by Dehornoy [25].

For Artin groups of FC type (free products of finite type groups with amalgamation over parabolic subgroups, for which the complete subgraphs of the labelled graph formed by deleting all ∞ -labelled edges from Γ are all of finite type), *asynchronously automatic* structures were constructed in [4], and

used to define quadratic time solutions to the word problem; we recall that an exponential (rather than quadratic) time solution is guaranteed by asynchronous automaticity. Right-angled Artin groups (those for which all the parameters m_{ij} are within the set $\{2, \infty\}$, which form a subset of FC type) were then proved automatic in [40, 74]. Very recently [50] Artin groups of FC type have been proved to be Helly, and hence biautomatic.

Mosher's paper [60] answered a major open question raised by Thurston's proof of the automaticity of the mapping class group of the punctured sphere. Using quite different techniques from Thurston, Mosher proved automaticity of the mapping class group of any surface of finite type, that is, the group of (orientation preserving) homeomorphisms modulo isotopy of any surface obtained from a compact surface by removing at most finitely many points. In the case of a surface with at least one puncture the automatic structure is explicitly defined (and could be constructed), in terms of a complex whose vertices are *ideal triangulations* on S (triangulations with vertex set the puncture set) and whose edges are elementary moves between ideal triangulations. The more general case can be reduced to the case of a punctured surface using a short exact sequence. The question of whether the mapping class group was in fact biautomatic was finally solved by Hamenstaedt's construction of a biautomatic structure in 2009 [38].

An Artin group is defined to have *large type* if all the associated parameters m_{ij} are at least 3, *extra large type* if all m_{ij} are at least 4. For large and especially extra large groups small cancellation techniques associated with negatively curved geometry were developed in [6]. All extra large Artin groups were proved biautomatic in [68], using those small cancellation techniques; the language is a set of geodesics over the standard generating set. All those groups and many others of large type were found by Brady and McCammond [11] to act appropriately on piecewise Euclidean non-positively curved 2-complexes of types A_2 or B_2 , and hence, by results of [34, 35] to be biautomatic (but in this case the biautomatic structure is defined over a non-standard generating set).

All Artin groups of large type were proved to be shortlex automatic over their standard generating sets in [44]. A rewrite system was described, which rewrote any word to shortlex geodesic form using sequences of moves on 2-generator substrings. The result extended beyond large type to *sufficiently large type*, where some parameters m_{ij} might take the value 2 (provided that for any triple i, j, k , if $m_{ij} = 2$, then either $m_{ik} = m_{jk} = 2$ or at least one of m_{ik} and m_{jk} is infinite). Biautomatic structures for all large type Artin groups (and in fact for the slightly large class of *almost large* groups) were proved to exist in [49], where all those groups were proved to have appropriate actions on *systolic* complexes. An Artin group is called almost large if for any triple

i, j, k it is only possible to have $m_{ij} = 2$ if one of m_{ik} or m_{jk} is infinite, and for any 4-set i, j, k, l at most 2 of $m_{ij}, m_{jk}, m_{kl}, m_{il}$ can be equal to 2 unless one of the four parameters is infinite.

5.2 Coxeter groups

The proof in [16] of shortlex automaticity of any Coxeter group relative to its standard generating set provided a result that had long been conjectured. We recall that a Coxeter group W is described by a presentation

$$\langle x_1, \dots, x_n \mid x_i^2 = 1, (x_i x_j)^{m_{ij}} = 1, i \neq j \in \{1, \dots, n\} \rangle,$$

relative to a Coxeter matrix (m_{ij}) and associated Coxeter diagram Γ ; the set $X = \{x_1, \dots, x_n\}$ is its *standard generating set*.

The proof of the theorem constructs an automatic structure for W using properties of its associated *root system*, which arises from the natural isomorphism between W and a reflection group \overline{W} as we now describe; more details can be found in [51]. The group \overline{W} is generated by a set of *reflections* r_1, \dots, r_n of \mathbb{R}^n defined by $r_i(v) := v - 2\langle v, e_i \rangle e_i$, for $v \in \mathbb{R}^n$, where $e_i : i = 1, \dots, n$ is a basis for \mathbb{R}^n and \langle, \rangle is the symmetric, bilinear form on \mathbb{R}^n defined by $\langle e_i, e_j \rangle = -\cos(\pi/m_{ij})$. The isomorphism from W to \overline{W} maps x_i to r_i , and induces an action of W on \mathbb{R}^n . The roots of W are defined to be the elements of the set $\Phi = W\{e_1, \dots, e_n\}$, which decomposes as a disjoint union $\Phi^+ \cup \Phi^-$ of *positive roots* (vectors $\sum \lambda_i e_i$ with all $\lambda_i \geq 0$) and their negatives.

Brink and Howlett's proof of regularity of the set of shortlex geodesic words in W is derived from their proof in [16] of the finiteness of the set of positive roots for W that *dominate* any given positive root; a positive root α is said to dominate a second positive root β if whenever $w(\alpha)$ is negative, for $w \in W$, then so is $w(\beta)$. We define $\tilde{\Delta}_W$ to be the set of positive roots that dominate no others. Then a word acceptor WA for a shortlex automatic structure for W can be built whose accepting states are all subsets of $\tilde{\Delta}_W$ [16, Proposition 3.3].

The transitions in WA are determined by the following observation from [16, Lemma 3.1]. When $w = x_{i_1} \cdots x_{i_l}$ is a shortlex geodesic word representing an element of W then, for $x_i \in X$, the word $w' = wx_i$ is non-geodesic if and only if there exists $j \in \{1, \dots, l\}$ for which $e_i = x_{i_l} \cdots x_{i_{j+1}}(e_{i_j})$. In the case where $w' = wx_i$ is geodesic, that fails to be shortlex minimal if and only if there exists $j \in \{1, \dots, l\}$ and a generator $x_k \prec x_{i_j}$ for which $e_i = x_{i_l} \cdots x_{i_j}(e_k)$. In that case the word $x_{i_1} \cdots x_{i_{j-1}} x_k x_{i_j} \cdots x_{i_l}$ is shortlex minimal. Based on these two facts, transition on a generator x_i from (the

state corresponding to) a subset S of $\tilde{\Delta}_W$ is to a failure state F if $e_i \in S$. But for $e_i \notin S$, transition is to the intersection with $\tilde{\Delta}_W$ of the set

$$S'' = \{x_i(\alpha) \mid \alpha \in S\} \cup \{e_i\} \cup \{x_i(e_k) \mid x_k \prec x_i\}.$$

A similar construction to the above, described in [47], proves regularity of the set of all geodesic words in G over S .

The question of whether all Coxeter groups are not just automatic but actually biautomatic remains open. Work of Niblo and Reeves [65] shows that any finitely generated Coxeter group G acts properly discontinuously by isometries on a locally finite, finite dimensional CAT(0) cube complex; their construction is based on the root system Φ associated with G , and an extension of the dominance relation of [16] from Φ^+ to Φ . When the action of G on the cube complex is cocompact, then biautomaticity follows, using [64]. Cocompact actions are proved in [65] to exist whenever G is right-angled or word hyperbolic (by [62] word hyperbolicity of G is recognisable from the diagram Γ). It is also observed in [65] that, by [76], cocompact actions are guaranteed whenever G contains only finitely many conjugacy classes of subgroups isomorphic to rank 3 parabolic subgroups $\langle x_i, x_j, x_k \rangle$ (associated with rank 3 subdiagrams Γ_{ijk} of Γ) for which m_{ij}, m_{ik}, m_{jk} are all finite; [21] used this result to derive biautomaticity of G provided that Γ contains no affine subdiagram of rank 3 or more. Subsequently, Caprace [20] proved biautomaticity of all relatively hyperbolic Coxeter groups using results from [48].

The dimension of a Coxeter group is defined to be the dimension of its *Davis complex*, equivalently the maximal rank of any of its spherical parabolic subgroups. It follows that a Coxeter group is 2-dimensional if none of the rank 3 subdiagrams Γ_{ijk} is spherical, equivalently if for all i, j, k , $\frac{1}{m_{ij}} + \frac{1}{m_{ik}} + \frac{1}{m_{jk}} \leq 1$. The biautomaticity of all 2-dimensional Coxeter groups is proved in [63]. The construction of a geodesic language generalises ideas from [65], and the result generalises an earlier result proving biautomaticity for certain 2-dimensional groups that used the results of [65].

6 Open problems

More than 30 years after the subject started there continue to be many open problems involving automatic groups. Some of these problems date from the beginning of the subject, and are listed in [26]. Some but not all of these have been mentioned within this chapter. In particular, it remains open whether automatic groups exist that are not biautomatic (see Section 2)

also whether automatic groups exist that do not have soluble conjugacy problem (see Section 2.2) , whether all soluble automatic groups must be virtually abelian. The most recent progress on this last question was made by the proof of Romankov [69], that a soluble biautomatic group must be virtually abelian (see Section 2.3.1).

It is still unknown whether a non-biautomatic Coxeter group can exist (Section 5.2), or a non-automatic Artin group (Section 5.1).

There are many open problems relating to group actions, in particular, whether a CAT(0) group must be automatic. The very recent construction in [56] of a 3-dimensional CAT(0) group that cannot be biautomatic (Section 4) represents a major advance on this problem; it does not resolve the question of automaticity. The question of whether biautomaticity or automaticity are implied for a 2-dimensional, piecewise Euclidean CAT(0) group remains open (but we note the recent contribution to this problem of the main result of [63], see Section 5.2). The 2-dimensional problem is number 43 on a list of open problems within geometric group theory that was published ten years ago in [32], and motivated a body of research, and rapid solution of some of the problems. However, some of the problems listed in this useful and extensive list, or in the earlier list [8], remain open.

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