Automaticity for graphs of groups

Susan Hermiller, Derek F. Holt, Sarah Rees and Tim Susse

15 May 2019

Abstract

In this article we construct asynchronous and sometimes synchronous automatic structures for amalgamated products and HNN extensions of groups that are strongly asynchronously (or synchronously) coset automatic with respect to the associated automatic subgroups, subject to further geometric conditions. These results are proved in the general context of fundamental groups of graphs of groups. The hypotheses of our closure results are satisfied in a variety of examples such as Artin groups of sufficiently large type, Coxeter groups, virtually abelian groups, and groups that are hyperbolic relative to virtually abelian subgroups.

2010 Mathematics Subject Classification: 20F65, 20F10, 20E06, 20F36
Key words: Automatic group, automatic coset system, graph of groups, relatively hyperbolic group, 3-manifold group, Artin group

1 Introduction

Closure properties for the classes of automatic and asynchronously automatic groups are known for a variety of group constructions. The class of automatic groups is closed with respect to finite index supergroups and subgroups, direct products, free products [16, Chapter 12], and graph products [17]. For amalgamated products and HNN extensions, closure for automaticity is also known in some special cases. Epstein et al. show in [16, Theorems 12.1.4, 12.1.9] that an amalgamated product or HNN extension of automatic groups along a finite subgroup is automatic, while Baumslag et al. [4] show closure for automaticity under amalgamated products with other technical restrictions. It is proved in particular in [4, Theorems E,B,D] that amalgamated products of two finitely generated free groups over a finitely generated subgroup are asynchronously automatic, and that amalgamated products of two finitely generated abelian groups over a subgroup or of two negatively curved groups over a cyclic subgroup are automatic.

This article derives automatic structures for new families of groups as a consequence of constructive proofs of closure properties for the class of groups with strong automatic coset systems. The study of groups that are automatic relative to a subgroup was introduced by Redfern in [26]. But Redfern had slightly weaker conditions on the associated structures than we need, and our definition of strong automatic coset systems comes from later work of Holt and Hurt in [19], where some fellow travelling conditions were added. With either variant of the definition, an automatic coset system provides a quadratic time algorithm for reducing an element of a coset of H in G to a normal form representative of that coset and, in particular, a quadratic time solution for the membership problem of elements of G in H.

Our article constructs strong automatic coset systems for fundamental groups of graphs of groups, given that the vertex groups have such systems with respect to corresponding edge groups, and given certain geometric conditions. In particular the construction can be applied to amalgamated products and HNN extensions, given those conditions. We build new automatic structures out of the automatic coset systems we are now able to build. Our results also generalise those of [16] relating to amalgamated products and HNN extensions.

The main results of the article are Theorems A, B, and D. The first two of these provide our most general combination theorems, using only conditions introduced in Section 2; the third deals with groups satisfying a particular condition on their geodesics. Those three main results are stated briefly at the end of this introduction, together with a pair of corollaries, relating specifically to the fundamental groups of compact 3-manifolds, and to Artin groups.

Related results on the construction of synchronous and asynchronous automatic structures on amalgamated products and HNN extensions were developed in [4], and their generalisations to graphs of groups were the topic of Shapiro's paper [29]. Some of our results are very similar to some of these, but others are distinct. The earlier papers [4] and [29] do not involve automatic coset systems, but their hypotheses are related to typical properties of automatic cosets systems (such as the subgroup H being quasiconvex in G), and their hypotheses are also related to our conditions of crossover and stability, which we discuss below. Another difference between their approach and ours is that their normal forms are defined using left cosets of the subgroups (so subgroup elements are situated at the right hand end of normal form words) whereas ours use right cosets. Since the definition of automaticity involves multiplying words by generators on the right, this difference is significant.

In Section 2 we give definitions and notation of basic concepts used throughout the article. The definitions of strong asynchronous and synchronous automatic coset systems for a group, subgroup pair (G, H) are given in Section 2.2. In that section we also prove Theorem 2.2, which constructs asynchronous or synchronous automatic structures for a group G, given a strong, asynchronous or synchronous, automatic coset system for (G, H) and an automatic structure, asynchronous or synchronous, for H. We can combine this result with our combination theorems for coset automatic systems to derive combination theorems for automatic structures.

Section 2.3 introduces the geometric conditions of limited crossover and stability that are used in our main results, and studies basic properties of these conditions.

The condition of limited crossover on a language L^H of coset representatives for H in G, with respect to a given generating set Y of H and a generating set Z of a possibly different subgroup of G, limits the Y-length of ugv^{-1} as a function of the Z-length of g when $g \in \langle Z \rangle$ and $ugv^{-1} \in H$, with $u, v \in L^H$. In our results about graphs of groups, we need to assume this condition when $\langle Y \rangle$ and $\langle Z \rangle$ are the edge subgroups of two edges with the same target vertex.

The condition of stability on an isomorphism ϕ between two groups H_1 and H_2 relates the lengths of an element $h \in H_1$ and its image $\phi(h)$ over specified generating sets of H_1 and H_2, respectively.

Section 3 is devoted to our first two main results for graphs of groups $\mathcal{G}$, Theorems A and B. These are the most general combination theorems of this article, building respectively strong asynchronous and strong synchronous coset systems for $\pi_1(\mathcal{G})$, given appropriate conditions of crossover and stabliity on vertex, edge subgroup pairs (and in the second case some further conditions). Necessary definitions and background on graphs of groups $\mathcal{G}$ are provided in Section 3.1, including a description of Higgins' [18] normal forms for $\pi_1(\mathcal{G})$. Section 3.2 contains the statement of Theorem A, together with asynchronous combination results Corollary 3.4 and Theorem 3.6 specifically for amalgamated products $G_1 *_H G_2$, and Corollary 3.5 specifically for HNN-extensions $G*_\phi$, of strongly asynchronously coset automatic group, subgroup pairs. The proof of Theorem A is given in Section 3.3. In Section 3.4, we prove Proposition 3.9 which derives strong synchronous coset automaticity from its asynchronous form, given a particular geometric condition. We apply this to derive Theorem B, our general closure result for graphs of groups that are strongly synchronously coset automatic.

The remainder of the article is devoted to finding applications of Theorems A and B. For some of these, such as Theorem D, some additional technical results are needed to derive them. In general, we find applications by providing proofs of strong asynchronous or synchronous coset automaticity, as well as crossover conditions and more, for various pairs (G, H). For many of our examples, we derive strong synchronous (rather than asynchronous) coset systems.

Section 4 is devoted to relatively hyperbolic groups; Section 4.1 contains definitions and a number of technical results that we need. Given a group G hyperbolic relative to a set of subgroups, and a specified such subgroup H, the technical result Proposition 4.5 provides conditions under which we can find a strong synchronous automatic coset system for (G, H) that satisfies crossover and some other conditions we need. Theorem 4.8 uses a combination of Proposition 4.5, and Theorems B and 2.2 to deduce strong synchronous coset automaticity relative to any peripheral subgroup of the fundamental group of a graph of relatively hyperbolic groups, under appropriate conditions on relevant subgroups and coset systems.

Results of Dahmani [12, Theorem 0.1] and Antolin and Ciobanu [3, Corollary 1.8] show that the fundamental group G of an acylindrical graph of finitely generated groups that are hyperbolic relative to abelian subgroups, in which all of the edge groups are peripheral, is automatic; in Corollary 4.10 we use Theorem 4.8 to give a new proof of this result. The special case of Corollary 4.10 in which the graph of groups arises from the JSJ decomposition of a 3-manifold yields Corollary C, which gives new automatic structures for fundamental groups of these 3-manifolds with respect to a Higgins language of normal forms (that is, normal forms derived from the JSJ composition). These fundamental groups were first shown to be automatic by Epstein et al. [16, Thm. 12.4.7] and Shapiro [29], but the structure of the associated languages is not transparent from the proofs; the results of Dahmani and of Antolin and Ciobanu provide a shortlex automatic structure.

Section 5 considers groups for which geodesics of a subgroup H of G 'concatenate up' to geodesics of G; such a pair (G, H) is also referred to in the literature [1, 2, 9] as an 'admissible pair'. We observe in Section 5.2 that this property holds for Coxeter groups and for Artin groups of sufficiently large type, relative to their parabolic subgroups. This property also holds for graph products of groups, relative to sub-graph products [10], [23, Prop. 14.4]. We apply Theorem A and Proposition 3.9 to deduce Theorem D. We note that we can apply this to prove automaticity of a variety of examples of Artin groups that were not previously known to be automatic; a family of such examples is described in Corollary E.

Finally, in Section 6 we derive several results involving abelian and virtually abelian groups. Proposition 6.1 establishes strong coset automaticity with limited crossover in finitely generated abelian groups. In Proposition 6.2 we prove that finitely generated virtually abelian groups G are strongly coset automatic with respect to any subgroup H; however, the question of whether any crossover conditions hold in this case remains open. We also prove assorted results on the strong synchronous coset automaticity of various types of amalgamated free products $G_1 *_H G_2$ for which (G_1, H) and (G_2, H) are both strongly coset automatic; in particular Proposition 6.5 proves the strongly synchronous coset automaticity of an amalgamated product of a finitely generated abelian group and a group that is hyperbolic relative to a collection of abelian subgroups, where amalgamation is over one of those subgroups.

1.1 Statement of main results

Each of these results refers to a graph of groups $\mathcal{G}(\Lambda) = (\Lambda = (V, \vec{E}), \{G_v : v \in V\}, \{G_e : e \in \vec{E}\}, \{\phi_e \mid e \in \vec{E}\})$ over a finite connected directed graph Λ. (For more details see Definitions 3.1 and 3.2 of Section 3.1.) We suppose that the groups G_v, G_e are finitely generated, with generating sets X_v and Y_e, respectively.

We refer the reader to Definitions 2.1, 2.3, 2.6 for the meanings of strong coset automaticity, limited crossover and stablility, respectively.

Theorem A *Let $\mathcal{G} = \mathcal{G}(\Lambda)$ be a graph of groups as above, and let e_0 be an edge of Λ. Suppose that the following conditions hold for each $e \in \vec{E}$.*

(i) The pair $(G_{\tau(e)}, G_e)$ is strongly asynchronously coset automatic with coset language $L^e_{\tau(e)} \subseteq (X^{\pm}_{\tau(e)})^*$ containing the empty word.

(ii) The triple $(G_e, G_{\overline{e}}, \phi_e)$ is stable with respect to $(Y_e, Y_{\overline{e}})$.

(iii) For each $f \in \vec{E}$ with $\tau(e) = \tau(f)$, the coset language $L^e_{\tau(e)}$ has limited crossover with respect to (Y_e, Y_f).

Then the pair $(\pi_1(\mathcal{G}), G_{e_0})$ is strongly asynchronously coset automatic.

Theorem B *Let $\mathcal{G} = \mathcal{G}(\Lambda)$ be a graph of groups as above, and let e_0 be an edge of Λ. Suppose that the following conditions hold for each $e \in \vec{E}$.*

(i) $Y_e \subseteq X_{\tau(e)}$.

(ii) The pair $(G_{\tau(e)}, G_e)$ is strongly synchronously coset automatic with coset language $L^e_{\tau(e)}$ satisfying $L^e_{\tau(e)} \subset \mathrm{GEO}(G_{\tau(e)}, X_{\tau(e)}) \cap [(X^{\pm}_{\tau(e)})^* \setminus Y^{\pm}_e (X^{\pm}_{\tau(e)})^*]$, the only representative in $L^e_{\tau(e)}$ of the identity coset is ϵ, and each element $g \in G_{\tau(e)}$ is represented by a word $y_g z_g \in \mathrm{GEO}(G_{\tau(e)}, X_{\tau(e)})$ with $y_g \in (Y^{\pm}_e)^*$ and $z_g \in L^e_{\tau(e)}$.

(iii) The triple $(G_e, G_{\overline{e}}, \phi_e)$ is 1–stable with respect to $(Y_e, Y_{\overline{e}})$.

(iv) For each $f \in \vec{E}$ with $\tau(e) = \tau(f)$, the coset language $L^e_{\tau(e)}$ has limited crossover with respect to (Y_e, Y_f).

Then the pair $(\pi_1(\mathcal{G}), G_{e_0})$ is strongly synchronously coset automatic.

Corollary C *Let M be an orientable, connected, compact 3-manifold with incompressible toral boundary whose prime factors have JSJ decompositions containing only hyperbolic pieces. Then the group $\pi_1(M)$ is automatic, with respect to a Higgins language of normal forms.*

Theorem D *Let $\mathcal{G} = \mathcal{G}(\Lambda)$ be a graph of groups as above. Suppose that the following conditions hold for each $e \in \vec{E}$.*

(i) $Y_e \subseteq X_{\tau(e)}$.

(ii) $\mathrm{GEO}(G_e, Y_e)$ *concatenates up to* $\mathrm{GEO}(G_{\tau(e)}, X_{\tau(e)})$.

(iii) The triple $(G_e, G_{\overline{e}}, \phi_e)$ is 1-stable with respect to $(Y_e, Y_{\overline{e}})$.

(iv) $G_{\tau(e)}$ is shortlex automatic with respect to an ordering of $X_{\tau(e)}$ in which all letters of $Y_e^{\pm}$ precede all letters of $X^{\pm}_{\tau(e)} \setminus Y_e^{\pm}$.

Let $\mathcal{L}$ be the set of coset languages $\mathrm{SL}^{G_e}_{G_{\tau(e)}}$, for $e \in \vec{E}$, and let $\mathcal{T}$ be any maximal tree in Λ. Then, for each $e_0 \in \vec{E}$, the pair $(\pi_1(\mathcal{G}), G_{e_0})$ is strongly synchronously coset automatic, with the Higgins coset language $L := L(\mathcal{G}, \mathcal{L}, e_0, \mathcal{T})$. Furthermore $L \subseteq \mathrm{GEO}^{G_{e_0}}$, and the group $\pi_1(\mathcal{G})$ is automatic.

Corollary E *Let Σ be a Coxeter graph of sufficiently large type. Given arbitrary subgraphs $\Lambda_1, \Lambda_2, \ldots, \Lambda_k$ of Σ, suppose that the Coxeter graph Σ' is formed by adjoining new vertices $v_1, v_2, \ldots, v_k$ to Σ together with the following edges from each v_i:*

to each vertex of Λ_i, with the label 2;
to each vertex of $\Sigma \setminus \Lambda_i$, with the label ∞;
to each vertex v_j with $j \neq i$, with the label ∞.

Then the Artin group $A_{\Sigma'}$ is automatic.

Acknowledgments

The first author was partially supported by grants from the National Science Foundation (DMS-1313559) and the Simons Foundation (Collaboration Grant number 581433).

2 Coset Automaticity and Crossover

2.1 Notation

Let G be a group. Throughout this article, all generating sets for all groups will be assumed to be finite. Let X be a finite generating set for G. We write $X^{\pm}$ for $X \cup X^{-1}$. We denote the length of a word $w \in X^{\pm}$ by $|w|$.

We denote by $\Gamma(G)$ (or $\Gamma(G,X)$ if it is necessary to specify X) the Cayley graph of G and let $d_{\Gamma(G)}$ be the path metric in $\Gamma(G)$. For each $g \in G$, we denote the length of a shortest word over $X^{\pm}$ that represents g by $|g|_X$, and call that the X-*length* of g. For any $g \in G$ and $w \in X^{\pm}$, let ${}_g w$ denote the path in $\Gamma(G)$ starting at the vertex g and labelled by the word w.

We write 1 for the identity element of G, and ϵ for the empty word in $(X^{\pm})^*$. For two words $w, x \in (X^{\pm})^*$, we write $w = x$ if w and x are the same word, and $w =_G x$ if w and x represent the same element of G.

2.2 Automatic coset systems and automatic structures

As before, let $G = \langle X \rangle$ with $|X| < \infty$. We define a *language for* G (over X) to be a set of words over $X^{\pm}$ that contains at least one representative of each element of G. Examples are provided by $\mathrm{GEO}(G,X) = \mathrm{GEO}$, the set of all words w over $X^{\pm}$ that are minimal length representatives of the elements of G they define, and $\mathrm{SL} \subseteq \mathrm{GEO}$, the set of all words w over $X^{\pm}$ that are minimal representatives of the elements they define with respect to the shortlex ordering (defined using some fixed ordering of $X^{\pm}$).

Let H be a finitely generated subgroup of G. A *coset language* for (G,H) is a set L^H (or L^H_G if it is necessary to specify G) of words over $X^{\pm}$ that contains at least one representative of each right coset Hg of H in G.

Examples of coset languages are provided by GEO^H (sometimes called GEO^H_G or even $\mathrm{GEO}^H_G(X)$), the set of all words w over X for which w is of minimal length as a representative of Hw, and $\mathrm{SL}^H \subseteq \mathrm{GEO}^H$ (sometimes called SL^H_G or $\mathrm{SL}^H_G(X)$), the set of all words w over X for which w is minimal with respect to the shortlex ordering as a representative of Hw (with respect to some fixed ordering of $X^{\pm}$).

Given a word w in $(X^{\pm})^*$ and a natural number t, let $w(t)$ denote the element of G represented by the prefix of w of length t; in the case that $t > |w|$, let $w(t) =_G w$. Two paths ${}_1 w$ and ${}_h w'$ in the Cayley graph $\Gamma(G,X)$ are said to *synchronously* K-*fellow travel* if for all $t \in \mathbb{N}$ we have $d_{\Gamma(G)}(w(t), hw'(t)) \leq K$. The paths ${}_1 w$ and ${}_h w'$ are said to *asynchronously* K-*fellow travel* if there exists nondecreasing surjective functions $\phi_1, \phi_2 : \mathbb{N} \to \mathbb{N}$ such that for all $t \in \mathbb{N}$ we have $d_{\Gamma(G)}(w(\phi_1(t)), hw'(\phi_2(t))) \leq K$.

Definition 2.1. A *strong asynchronous automatic coset system* for (G,H) is defined to be a coset language $L^H \subseteq X^{\pm}$ together with a constant K, such that

(i) L^H is a regular language (that is, the language of a finite state automaton),

(ii) if $v, w \in L^H$ and $h \in H$ with $d_{\Gamma(G,X)}(v, hw) \leq 1$, then the paths ${}_1 v$ and ${}_h w$ in $\Gamma(G)$ asynchronously K-fellow travel. (So, in particular, we have $|h|_X \leq K$.)

If (G,H) has a strong asynchronous automatic coset system as above, then we say that (G,H) is *strongly asynchronously coset automatic* (or SACA for short), with *coset language* L^H, and *fellow traveller constant* K. If the fellow traveller condition above can be replaced by a synchronous fellow traveller condition, then we say that G is *strongly synchronously coset automatic* (or SSCA) or just *strongly coset automatic*.

We note that our definition of SSCA matches the definition of coset automaticity in [19]. Moreover, in the case that the subgroup H is the trivial group, the definition of SSCA is equivalent to the definition of automatic [16]. We refer the reader to [16] or [20] for further information on fellow traveller properties, regular languages, finite state automata, and automatic groups.

The following result allows us to construct automatic structures for groups from automatic coset systems.

Theorem 2.2. *Let $G = \langle X \rangle$ be a group, and $H = \langle Y \rangle$ a subgroup of G. Suppose that (G, H) is strongly asynchronously coset automatic with language L^H with respect to X, and H is asynchronously automatic with language L_H with respect to Y. Then*

(i) the group G is asynchronously automatic over $X \cup Y$, with language $L := L_H L^H$ (the concatenation of L_H and L^H);

(ii) if (G, H) is strongly coset automatic and H is automatic (that is, both structures are synchronous), then G is automatic. Furthermore, if L_H and L^H are both synchronous structures and L_H contains only finitely many representatives of each element of H, then the language $L := L_H L^H$ is the language of a synchronous automatic structure for G.

Note that we do not require X, Y to be disjoint.

Proof. Suppose first that L_H and L^H satisfy the asynchronous K–fellow traveller property. Since regular languages are closed under concatenation, the language L is regular. We shall verify an asynchronous fellow traveller property for L with constant κK^2, where κ is the maximum X-length of any $y \in Y$.

Suppose that wv, $w'v'$ are words in L with $w, w' \in L_H$ and $v, v' \in L^H$. First suppose that $wv =_G w'v'$. In this case v and v' represent the same coset of H in G, and so pair of paths ${}_1v$ and ${}_{w^{-1}w'}v'$, and hence also the pair ${}_wv$ and ${}_{w'}v'$, asynchronously K-fellow travel. In particular $d_{\Gamma(G)}(w, w') \le K$, and hence, applying the fellow property for L_H we deduce that w, w' asynchronously fellow travel at distance K^2.

Similarly, if $x \in X$ and $wvx =_G w'v'$, then the same argument shows that the paths ${}_wv$ and ${}_{w'}v'$ K-fellow travel, and w and w' K^2-fellow travel.

Finally, suppose that $y \in Y$, and that $wvy =_G w'v'$. Writing $y = x_1 \cdots x_\kappa$ with each $x_i \in X^\pm$, we have that the paths ${}_wv$ and ${}_{w'}v'$ asynchronously κK–fellow travel, and so w and w' fellow travel at distance κK^2.

In all cases the paths ${}_1w$ and ${}_1w'$ asynchronously κK^2–fellow travel, and the paths ${}_wv$ and ${}_{w'}v'$ asynchronously κK–fellow travel. Thus, the paths ${}_1wv$ and ${}_1w'v'$ asynchronously κK^2–fellow travel, as desired. This proves (i).

To prove (ii), suppose that L_H and L^H are the languages of synchronous structures. If L_H contains infinitely many representatives of some elements of H then, by [16, Thm 2.5.1], we can replace it by a language consisting of unique representatives. So, in proving the first assertion of (ii), we may assume that L_H contains only finitely many such representatives. Let K be a synchronous fellow traveller constant for both structures. Then by [16, Thm 2.3.9], there is a constant N such that whenever $u, u' \in L_H$ and the paths ${}_1u$ and ${}_1u'$ end a distance at most 1 apart, the lengths of the words u and u' differ by at most N.

Let $w, w' \in L_H$ and $v, v' \in L^H$ satisfy $wvx =_G w'v'$ for some $x \in X \cup Y \cup \{\epsilon\}$. As in the proof above, we see that the paths ${}_wv$, ${}_{w'}v'$ synchronously κK–fellow travel and so the paths ${}_1w$ and ${}_1w'$ end at distance at most κK apart in $\Gamma(G, X)$. Hence the paths ${}_1w$ and ${}_1w'$ synchronously κK^2–fellow travel, and their lengths differ by at most κKN. Thus, the paths ${}_1wv$ and ${}_1w'v'$ synchronously $\kappa K^2 + \kappa KN$–fellow travel. $\square$

2.3 Crossover and stability

The properties of crossover and stability for coset systems are fundamental for us in order to prove the results of Section 3.

Definition 2.3. Let Y and Z be finite subsets of a group G, and let $H = \langle Y \rangle$. Let $1 \le \lambda \in \mathbb{N}$. We say that the coset language L^H for (G, H) has λ-*limited crossover* with respect to (Y, Z) if, for any $g \in \langle Z \rangle$ with $|g|_Z \le \lambda$, and any $u, v \in L^H$ with $ug \in Hv$, we have $|ugv^{-1}|_Y \le \lambda$. We say that L^H has *limited crossover* with respect to (Y, Z) if it has λ-limited crossover for some λ. If $Y = Z$, we use the term limited crossover *with respect to* Y.

As we shall show in Sections 4, 5, and 6, the limited crossover property is satisfied, for example, by Coxeter groups and by Artin groups of large type where Y and Z are arbitrary subsets of the standard generating sets, by finitely generated abelian groups with $Y = Z$ an arbitrary finite subset, and by groups that are relatively hyperbolic with respect to virtually abelian or hyperbolic parabolic subgroups.

The following result will be useful for finding common crossover constants for a collection of languages and generating sets.

Lemma 2.4. *Suppose that for finite subsets Y, Z of a group $G = \langle X \rangle$, $H = \langle Y \rangle$, $1 \le \lambda \in \mathbb{N}$, and coset language $L^H \subseteq (X^{\pm})^*$ for (G, H), the set L^H has λ-limited crossover with respect to (Y, Z). Then L^H has $k\lambda$-limited crossover with respect to (Y, Z), for any $k \in \mathbb{N}$.*

Proof. Suppose that $g \in \langle Z \rangle$ with $|g|_Z \le k\lambda$ and that $u, v \in L^H$ satisfy $ug \in Hv$. Then we can decompose g as a product $g = g_1 \cdots g_k$ of elements $g_i \in \langle Z \rangle$, with each $|g_i|_Z \le \lambda$. We choose $v_1, \ldots, v_k \in L^H$ such that, for each i, $ug_1 \cdots g_i \in Hv_i$; in particular we choose $v_k = v$. Then we have $ug_1 \in Hv_1$, and for each $i = 2, \ldots, k$ $v_{i-1}g_i \in Hv_i$. We deduce from the limited crossover condition that ug_1v^{-1} and all of the elements $v_{i-1}g_iv_i^{-1}$ for $2 \le i \le k$ have Y-length at most λ. The product of these elements has Y-length at most $k\lambda$ and is equal to ugv^{-1}. $\square$

The following stronger version of crossover leads to a variant of the result of Corollary 3.4 for amalgamated products, proved in Theorem 3.6.

Definition 2.5. Let Y, Z be finite subsets of G, $H = \langle Y \rangle$, and $1 \le \lambda \in \mathbb{N}$. We say that a coset language L^H for (G, H) has λ-*maximal crossover* with respect to (Y, Z) if, for any $g \in \langle Z \rangle$ and any $u, v \in L^H$ with $u \notin H$ and $ug \in Hv$, we have $|ugv^{-1}|_Y \le \lambda$. If $Y = Z$, then we use the term λ-maximal crossover *with respect to* Y. We say that L^H has *maximal crossover* with respect to the subgroup $\langle Z \rangle$ of G if L^H has λ-maximal crossover with respect to (Y', Z) for some generating set Y' of H and $1 \le \lambda \in N$.

We note that the maximal crossover property would not hold in the case that H and G/H are infinite if we did not impose the condition $u \notin H$, but we do not need that condition in the definition of limited crossover. It is straightforward to show that if the maximal crossover property holds for some finite generating set Y of H, then it holds (but probably with a different parameter λ) for any other finite generating set. Further, for some finite generating set of H, L^H has 1-maximal crossover.

This stronger property of maximal crossover is unusual but, as we shall show in Section 4.2, it holds for groups that are hyperbolic relative to a collection of finitely generated groups that are either virtually abelian or hyperbolic, where Y and Z are suitably chosen generating sets of two of the parabolic subgroups.

Definition 2.6. Suppose that $H_1 = \langle Y_1 \rangle$ and $H_2 = \langle Y_2 \rangle$ are isomorphic groups with isomorphism $\phi : H_1 \to H_2$. We say that (H_1, H_2, ϕ) is *μ-stable* with respect to (Y_1, Y_2) if, whenever $h \in H_1$ with $|h|_{Y_1} \leq \mu$, we have $|\phi(h)|_{Y_2} \leq \mu$. Provided that each of H_1, H_2 is associated with just one generating set, we may omit the phrase'with respect to (Y_1, Y_2)', and in general we shall do that. We say that (H_1, H_2, ϕ) is *stable* if it is μ-stable for some μ.

We have the following results.

Lemma 2.7. *For groups $H_1 = \langle Y_1 \rangle, H_2 = \langle Y_2 \rangle$, related by an isomorphism $\phi : H_1 \to H_2$ of G, if (H_1, H_2, ϕ) is μ-stable, then it is also $k\mu$-stable for any $k \in \mathbb{N}$.*

We omit the proof of this, which is nearly identical to the proof of Lemma 2.4.

Lemma 2.8. *(i) Let X_1 and X_2 be finite generating sets for a group G, and let L_1^H be a coset language for the pair (G, H) with respect to X_1. Then there is a coset language L_2^H for (G, H) with respect to X_2, such that the subset of G represented by the words in L_2^H is the same as for L_1^H and such that, if any of the properties* SACA*,* SSCA*, limited crossover, or maximal crossover hold in L_1^H, then they also hold in L_2^H.*

(ii) If $H \leq F \leq G$ with $|G : F|$ finite, and (G, H) has a coset language L_G^H satisfying any of SACA*,* SSCA*, or limited crossover or maximal crossover relative to a pair of finite generating sets for H, then there is a coset language L_F^H for (F, H) with the same properties. Furthermore, we can choose L_F^H such that the subset of the group that it represents is the intersection with F of the subset represented by L_G^H.*

(iii) If $H \leq G \leq F$ with $|F : G|$ finite and (G, H) is strongly asynchronously (resp. synchronously) coset automatic, then so is (F, H).

We note that in (iii) it is not clear that either of the crossover properties is preserved.

Proof. (i) Let $\pi_1 \colon X_1^* \to G$ and $\pi_2 \colon X_2^* \to G$ be the natural projection maps. For each generator $x \in X_1$ choose a word $w(x) \in X_2^*$ so that $\pi_1(x) = \pi_2(w(x))$, and let let $\varphi \colon X_1^* \to X_2^*$ be the corresponding semigroup homomorphism. Then we imitate the construction of L_2^H from $\varphi(L_1^H)$ exactly as in [16, Theorem 2.4.1] for automatic structures, and then, by the same argument as in [16], L_2^H is SACA or SSCA if L_1^H is. Furthermore, we have $\pi_2(L_2^H) = \pi_1(L_1^H)$.

Now let $Y \subseteq H$ be a finite generating set for H, and $Z \subseteq G$ be any finite subset. Suppose that $L_1^H \subseteq X_1^*$ has λ–limited (resp. λ–maximal) crossover with respect to (Y, Z), Since both the limited and maximal crossover properties depend only on the image of the language in G and the generating set of H, it follows that L_2^H has λ–limited (resp. λ–maximal) crossover with respect to (Y, Z), as desired.

We omit the proofs of (ii) and (iii), which are straightforward adaptations of the proof of [16, Theorem 4.1.4]. $\square$

Next we note that any coset language containing the empty word can have all other representatives of the same coset removed, without altering SACA or crossover conditions.

Lemma 2.9. *Suppose that the group $G = \langle X \rangle$ is* SACA *with language L^H containing the empty word ϵ, with respect to a finitely generated subgroup $H = \langle Y \rangle$. Suppose also that $Z_1, ..., Z_m$ are finitely many finite subsets of G such that L^H has limited crossover with respect to (Y, Z_i) for all i. Let $\widetilde{L}^H := L^H \setminus S$, where S is the set of nonempty words in L^H that represent the identity coset H in G. Then the pair (G, H) is* SACA *with language $\widetilde{L}^H$, and $\widetilde{L}^H$ has limited crossover with respect to (Y, Z_i) for all i as well.*

3 Automatic structures for graphs of groups

Our goal in this section is to prove Theorem A, that free products with amalgamation, HNN extensions, and more generally fundamental groups of graphs of groups of asynchronously automatic groups with well-behaved coset automatic structures, are also asynchronously automatic; the proof is given in Section 3.3. The resulting structure is asynchronous, but under certain circumstances a strong asynchronous coset system contains a synchronous system as a substructure, as is proved in Proposition 3.9. We apply the proposition to deduce a synchronous closure result Theorem B for graphs of groups with particular conditions on associated coset automatic structures.

We begin this section with definitions and notation for graphs of groups.

3.1 Background on graphs of groups and Higgins normal forms

For a directed graph Λ with vertex set V and directed edge set $\vec{E}$ (written $\Lambda = (V, \vec{E})$), we denote the initial and terminal vertices of an edge $e \in \vec{E}$ by $\iota(e)$ and $\tau(e)$ respectively. We assume that associated with each edge $e \in \vec{E}$, there is an oppositely oriented edge $\overline{e}$ with $\iota(\overline{e}) = \tau(e)$ and $\tau(\overline{e}) = \iota(e)$. We define $P(\Lambda)$ to be the set of directed paths of Λ, and where $p = e_1 \cdots e_k \in P(\Lambda)$, we define $\iota(p) = \iota(e_1)$, $\tau(p) = \tau(e_k)$.

Definition 3.1. A *graph of groups* is a quadruple $\mathcal{G} = (\Lambda, \{G_v \mid v \in V\}, \{G_e \mid e \in \vec{E}\}, \{\phi_e \mid e \in \vec{E}\})$, where $\Lambda = (V, \vec{E})$ is a directed graph, each G_v is a group, and for each $e \in \vec{E}$, G_e is a subgroup of $G_{\tau(e)}$, $\phi_e : G_e \to G_{\overline{e}}$ is an isomorphism, and $\phi_e^{-1} = \phi_{\overline{e}}$.

We call the subgroups G_v, G_e the *vertex* and *edge* groups of $\mathcal{G}$, respectively.

Following standard practice, we assume that the G_v have pairwise trivial intersections. Whenever we refer to a graph of groups $\mathcal{G} = \mathcal{G}(\Lambda)$ in this article we will use the notation of Definition 3.1. In addition we will use the notation X_v, Y_e for specified generating sets for G_v, G_e, respectively.

Definition 3.2. Let $\mathcal{G} = (\Lambda, \{G_v \mid v \in V\}, \{G_e \mid e \in \vec{E}\}, \{\phi_e \mid e \in \vec{E}\})$ be a graph of groups with connected graph Λ, and let $\mathcal{T}$ be a maximal tree in Λ. The *fundamental group of* $\mathcal{G}$ *at* $\mathcal{T}$, denoted $\pi_1(\mathcal{G}) = \pi_1(\mathcal{G}, \mathcal{T})$, is the group generated by the disjoint union of all of the groups G_v and the set of symbols $\{s_e \mid e \in \vec{E}\}$, subject to the relations:

(i) $s_{\overline{e}} = s_e^{-1}$ for all $e \in \vec{E}$,
(ii) $s_e = 1$ for all directed edges e in $\mathcal{T}$, and
(iii) $s_e g s_e^{-1} = \phi_e(g)$ for all $e \in \vec{E}$ and $g \in G_e$.

When Λ consists of two vertices joined by an edge, or of a single vertex together with a loop, then the fundamental group is a free product with amalgamation or HNN-extension, respectively. We refer the reader to [27, 28] for basic facts about graphs of groups. In particular, up to isomorphism, $\pi_1(\mathcal{G}, \mathcal{T})$ does not depend on the choice of the maximal tree $\mathcal{T}$.

Now we provide a description of the language for $\pi_1(\mathcal{G})$ that we use in our proof of Theorem A. This is a set of words representing normal forms provided by Higgins in [18], but modified to work with right rather than left cosets, and to provide words over generating sets rather than normal forms that are products of elements; see [8, Prop. 3.3] for more details.

For each $v \in V$, let X_v be a finite generating set for the vertex group G_v; note that the X_v are pairwise disjoint. We consider the generating sets

$$\widehat{X} := \bigcup_{v \in V} X_v \cup \{s_e : e \in \vec{E}\} \qquad \text{and} \qquad X := \bigcup_{v \in V} X_v \cup \{s_e : e \in \vec{E} \setminus \vec{E}_{\mathcal{T}}\}$$

for $\pi_1(\mathcal{G})$. For any product $w \in (\widehat{X}^{\pm})^*$, we define its *deflation* $\mathrm{Defl}(w, \mathcal{T}) \in (X^{\pm})^*$ to be the word derived from w by omitting from it every s_e for which e is in the set $\vec{E}_{\mathcal{T}}$ of directed edges of the tree $\mathcal{T}$.

Choose a vertex $v_0 \in V$, and let $L_{v_0} \subset (X_{v_0}^{\pm})^*$ be a language for G_{v_0}. For each $e \in \vec{E}$, let $L^e_{\tau(e)} \subset (X^{\pm}_{\tau(e)})^*$ be a coset language for $(G_{\tau(e)}, G_e)$, and let $\mathcal{L}$ be the collection of languages $\{L^e_{\tau(e)}\}$.

Now define $\widehat{L}(\mathcal{G}, \mathcal{L}, L_{v_0}, v_0, \mathcal{T}) \subset (\widehat{X}^{\pm})^*$ to be the set of all words of the form:

$$w_0 s_{e_1} u_1 \cdots s_{e_k} u_k,$$

where

(i) $p = e_1 \cdots e_k \in P(\Lambda)$ with $\iota(p) = v_0$;
(ii) $w_0 \in L_{v_0}$ and $u_i \in L^{e_i}_{\tau(e_i)}$ for $1 \leq i \leq k$;
(iii) if $e_{i+1} = \overline{e}_i$, then u_i does not represent an element of G_{e_i};
(iv) if $k > 0$ and $e_k \in \vec{E}_{\mathcal{T}}$, then u_k does not represent an element of G_{e_k}.

Then every element of $\pi_1(\mathcal{G})$ has at least one representative of this form. We refer to this set as the *inflated Higgins language* for the group $\pi_1(\mathcal{G}, \mathcal{T})$ with respect to the triple $(\mathcal{L}, v_0, L_{v_0})$. The language

$$L(\mathcal{G}, \mathcal{L}, L_{v_0}, v_0, \mathcal{T}) := \{\mathrm{Defl}(w, \mathcal{T}) \mid w \in \widehat{L}(\mathcal{G}, \mathcal{L}, L_{v_0}, v_0, \mathcal{T})\}$$

over X is the associated *Higgins language* for $\pi_1(\mathcal{G})$.

Next suppose that e_0 is any directed edge in Λ. We define $\widehat{L}(\mathcal{G}, \mathcal{L}, e_0, \mathcal{T})$ to be the of all words over $\widehat{X}$ of the from of the form

$$u_0 s_{e_1} u_1 \cdots s_{e_k} u_k$$

where $u_0 \in L^{e_0}_{\tau(e_0)}$ and the conditions (i)–(iv) above hold with $v_0 = \tau(e_0)$. Then each coset of G_e in $\pi_1(\mathcal{G})$ has at least one representative in this language. we call this the *inflated Higgins coset language* for the pair $(\pi_1(\mathcal{G}, \mathcal{T}), G_{e_0})$ with respect to $(\mathcal{L}, e_0)$. Similarly, the language

$$L(\mathcal{G}, \mathcal{L}, e_0, \mathcal{T}) := \{\mathrm{Defl}(w, \mathcal{T}) \mid w \in \widehat{L}(\mathcal{G}, \mathcal{L}, e_0, \mathcal{T})\}$$

over X is the associated *Higgins coset language* for $(\pi_1(\mathcal{G}), G_{e_0})$.

Remark 3.3. We note that in the case when L_{v_0} and $L^e_{\tau(e)}$ are languages with unique representatives of G_{v_0} and of cosets of G_e in $G_{\tau(e)}$, respectively, then the corresponding Higgins languages are sets with unique representatives of $\pi_1(\mathcal{G})$ and cosets of G_{e_0} in $\pi_1(\mathcal{G})$, respectively.

3.2 Strong asynchronous automatic coset systems for graphs of groups

This section is devoted to the statement of our main graph of groups result Theorem A, together with Corollaries 3.4 and 3.5, for amalgamated products and HNN extensions, which are special cases of this. We defer the proof of the theorem to the following section, Section 3.3. We conclude this section with a variant of the result for amalgamated free products, in the case that one group has maximal crossover.

Theorem A. *Let $\mathcal{G} = (\Lambda = (V, \vec{E}), \{G_v : v \in V\}, \{G_e : e \in \vec{E}\}, \{\phi_e \mid e \in \vec{E}\})$ be a graph of groups over a finite connected directed graph Λ with an edge e_0. Let X_v and Y_e be finite generating sets of the groups G_v and G_e, respectively. Suppose that the following conditions hold for each $e \in \vec{E}$.*

(i) The pair $(G_{\tau(e)}, G_e)$ is strongly asynchronously coset automatic with coset language $L^e_{\tau(e)} \subseteq (X^{\pm}_{\tau(e)})^*$ containing the empty word ϵ.

(ii) The triple $(G_e, G_{\overline{e}}, \phi_e)$ is stable with respect to $(Y_e, Y_{\overline{e}})$.

(iii) For each $f \in \vec{E}$ with $\tau(e) = \tau(f)$, the coset language $L^e_{\tau(e)}$ has limited crossover with respect to (Y_e, Y_f).

Then the pair $(\pi_1(\mathcal{G}), G_{e_0})$ is strongly asynchronously coset automatic.

In the special cases of amalgamated products and HNN extensions, we immediately obtain the following results as corollaries of the above result together with Theorem 2.2.

Corollary 3.4 (Amalgamated products). *Let G_1 and G_2 be groups with a common subgroup $H = G_1 \cap G_2$, and suppose that G_1, G_2 and H are all finitely generated. Suppose that the pairs (G_1, H) and (G_2, H) are both strongly asynchronously coset automatic and that, for some finite generating set Y of H, each of the associated coset languages has limited crossover with respect to (Y, Y). Then $(G_1 *_H G_2, H)$ has a strong asynchronous automatic coset system. Moreover, if the group H is asynchronously automatic, then so is the amalgamated product $G_1 *_H G_2$.*

Corollary 3.5 (HNN extensions). *Let G, H_1, H_2 be finitely generated with $H_1, H_2 \leq G$ and $\phi\colon H_1 \to H_2$ an isomorphism. Further, let $H_i = \langle Y_i \rangle$ for $|Y_i| < \infty$, for $i = 1, 2$. Suppose that:*

(i) the pairs (G, H_j) are strongly asynchronously coset automatic with coset language L^{H_j} for $j = 1, 2$;

(ii) L^{H_j} has limited crossover with respect to (Y_i, Y_j) for each of i, j in $\{1, 2\}$; and

(iii) the triples (H_1, H_2, ϕ) and (H_2, H_1, ϕ^{-1}) are stable.

Then $(G*_\phi, H_i)$ is strongly asynchronously coset automatic for $i = 1, 2$. Moreover, if the (isomorphic) groups H_i are asynchronously automatic, then so is the HNN extension G_1*_ϕ.

In the presence of maximal crossover, the following variation of Theorem A holds for amalgamated free products. The proof is analogous to the proof of Theorem A in Section 3.3, although maximal crossover allows us to simplify the argument somewhat.

Theorem 3.6. *Let $H \leq G_1 \cap G_2$ be a finitely generated group within the intersection of groups $G_1 = \langle X_1 \rangle$, $G_2 = \langle X_2 \rangle$, Suppose that (G_1, H) and (G_2, H) are both strongly asynchronously coset automatic, and that the language for (G_1, H) has maximal crossover. Then $(G_1 *_H G_2, H)$ is strongly asynchronously coset automatic.*

One situation in which we could apply this result is when G_1 is hyperbolic relative to a collection of virtually abelian groups with H a peripheral subgroup (cf Proposition 4.5) and when H is an arbitrary subgroup of the virtually abelian group G_2 (cf Proposition 6.2).

3.3 Proving Theorem A

In order to prove Theorem A, we need to define a procedure that we call *cascading*, that will convert a given word of a particular form over $\widehat{X}^{\pm}$ into another word representing the same group element, which (as we shall show in Lemma 3.8) is in the inflated Higgins language.

Definition 3.7. Let $\mathcal{G} = (\Lambda = (V, \vec{E}), \{G_v = \langle X_v \rangle\}, \{G_e\}, \{\phi_e\})$ be a graph of groups, and let $v_0 \in V$.

Let $L_{v_0} \subset (X_v^{\pm})^*$ be a language for G_{v_0} and let $\mathcal{L} = \{L^e_{\tau(e)} : e \in \vec{E}\}$ be a set of coset languages for the pairs $(G_{\tau(e)}, G_e)$, for which each $L^e_{\tau(e)}$ is a language over $X_{\tau(e)}$, and for which the only representative in each $L^e_{\tau(e)}$ of the identity coset G_e is the empty word.

Let $w = w_0 s_{e_1} w_1 \cdots s_{e_k} w_k \in (\widehat{X}^{\pm})^*$, where $p = e_1 \cdots e_k \in P(\Lambda)$ with $\iota(p) = v_0$, $w_0 \in (X^{\pm}_{v_0})^*$, and $w_i \in (X^{\pm}_{\tau(e_i)})^*$ for $i = 1, \ldots, k$.

An $(\mathcal{L}, L_{v_0})$*–cascade* of w is a word $u \in (\widehat{X}^{\pm})^*$ satisfying $u =_{\pi_1(\mathcal{G})} w$ that is obtained as follows.

(i) Select $u_k \in L^{e_k}_{\tau(e_k)}$ with $w_k =_{G_{\tau(e_k)}} h_k u_k$ for some $h_k \in G_{e_k}$.

(ii) For $j = k-1, \ldots, 1$, select $u_j \in L^{e_j}_{\tau(e_j}$, with $w_j \phi_{e_{j+1}}(h_{j+1}) =_{G_{\tau(e_j)}} h_j u_j$ for some $h_j \in G_{e_j}$.

(iii) Select $u_0 \in L_{v_0}$ representing the element $w_0 \phi_{e_1}(h_1)$ in G_{v_0}.

(iv) Remove from $u_0 s_{e_1} u_1 \cdots s_{e_k} u_k$ the maximal suffix of the form $s_{e_j} s_{e_{j+1}} \cdots s_{e_k}$ for which $e_i \in \vec{E}_{\mathcal{T}}$ for all $j \le i \le k$, to obtain u.

The proof of the following lemma is basically the proof in [8, Proposition 3.4].

Lemma 3.8. *Let $\mathcal{G}$ be a graph of groups over a finite connected graph $\Lambda = (V, \vec{E})$, and assume the notation of Section 3.1. Let $\widehat{L} = \widehat{L}(\mathcal{G}, \mathcal{L}, L_{v_0}, v_0, \mathcal{T})$ be an inflated Higgins language over $\widehat{X} = \cup_{v \in V} X_v \cup \{s_e : e \in \vec{E}\}$ for which the only representative in each $L^e_{\tau(e)}$ of the identity coset G_e is the empty word. Let $w = w_0 s_{e_1} w_1 \cdots s_{e_k} w_k$ be a word over $\widehat{X}$ as in Definition 3.7, and suppose that for all $1 \le i \le k-1$ either $e_{i+1} \ne \overline{e}_i$ or w_i does not represent an element of G_{e_i}.*

(i) If u is an $(\mathcal{L}, L_{v_0})$–cascade of w, then $u \in \widehat{L}$.

(ii) Suppose that if both $k > 0$ and $e_k \in \vec{E}_{\mathcal{T}}$ then w_k does not represent an element of G_{e_k}. Then any $w' \in \widehat{L}$ with $w' =_{\pi_1(\mathcal{G})} w$ is an $(\mathcal{L}, L_{v_0})$–cascade of w of the form $w' = w'_0 s_{e_1} w'_1 \cdots s_{e_k} w'_k$; that is, the paths in Λ associated with w and w' are the same, and there exist elements $h_i \in G_{e_i}$ for $1 \le i \le k$ such that $w_k =_{G_{\tau(e_k)}} h_k w'_k$, and for $k > i \ge 1$, $w_i \phi_{e_{i+1}}(h_{i+1}) =_{G_{\tau(e_i)}} h_i w'_i$, and $w_0 \phi_{e_1}(h_1) =_{G_{v_0}} w'_0$.

Proof. An $(\mathcal{L}, L_{v_0})$–cascade word u of w is obtained by removing a suffix $s_{e_j} s_{e_{j+1}} \cdots s_{e_k}$ of letters corresponding to edges in the tree $\mathcal{T}$ from a word of the form $u_0 s_{e_1} u_1 \cdots s_{e_k} u_k$. For each index ℓ, we have $w_\ell =_{G_{\tau(e_\ell)}} h_\ell u_\ell \phi_{e_{\ell+1}}(h_{\ell+1})^{-1}$ and, if $e_{\ell+1} = \overline{e}_\ell$, then $h_\ell, \phi_{e_{\ell+1}}(h_{\ell+1}) \in G_{e_\ell}$ and w_ℓ does not represent an element of G_{e_ℓ}, so the word u_ℓ also cannot represent an element of G_{e_ℓ}. Moreover, since the only representative of the identity coset in each of the coset languages is the empty word, then after removing the maximal suffix of letters associated with edges in $\mathcal{T}$ from the word $u_0 s_{e_1} u_1 \cdots s_{e_k} u_k$, either the resulting word u is in L_{v_0}, or u ends with a letter $s_{e_{j-1}}$ with $e_{j-1} \notin \vec{E}_{\mathcal{T}}$, or u ends with a word $s_{e_{j-1}} u_{j-1}$ in which u_{j-1} does not represent the identity coset. Hence u is in the inflated Higgins set $\widehat{L}$.

Suppose further that the additional hypothesis of (ii) holds. If $k > 0$ and $e_k \in \vec{E}_{\mathcal{T}}$ then, since $w_k =_{G_{\tau(e_k)}} h_k u_k$ with $h_k \in G_{e_k}$, and w_k does not represent an element of G_{e_k}, again we see that the word u_k cannot represent an element of G_{e_k}. Hence in any case no suffix of letters is removed in the last step of the cascade procedure, and the $(\mathcal{L}, L_{v_0})$–cascade u of w has the form $u = u_0 s_{e_1} u_1 \cdots s_{e_k} u_k$ with the same associated path in Λ as w.

Now let $\widetilde{L}_{v_0} \subseteq (X^{\pm}_{v_0})^*$ be a set of unique representatives for the elements of G_{v_0} containing the empty word ϵ, and for each $e \in \vec{E}$ let $\widetilde{L}^e \subseteq L^e_{\tau(e)}$ be a set of unique representatives of the right cosets of G_e in $G_{\tau(e)}$, containing ϵ. Let $\widetilde{L} := \widehat{L}(\mathcal{G}, \{\widetilde{L}^e\}, \widetilde{L}_{v_0}, v_0, \mathcal{T})$ be the associated inflated Higgins language. By Remark 3.3, each element of $\pi_1(\mathcal{G})$ is represented by a unique element of $\widetilde{L}$.

Let $\widetilde{w}$ and $\widetilde{w}'$ be $(\{\widetilde{L}^e\}, \widetilde{L}_{v_0})$–cascades of w and w', respectively. The words w and w' both satisfy the hypotheses in (ii), and so the proof above shows that $\widetilde{w}, \widetilde{w}' \in \widetilde{L}$. Now $\widetilde{w} =_{\pi_1(\mathcal{G})} w =_{\pi_1(\mathcal{G})} w' =_{\pi_1(\mathcal{G})} \widetilde{w}'$, and so the uniqueness of representatives in $\widetilde{L}$ implies that $\widetilde{w}$ and $\widetilde{w}'$ are the same word over $\widetilde{X}$. Moreover, the argument above shows that the paths in Λ associated with w, $\widetilde{w}$, and w'

are the same. In particular, we can write $\widetilde{w} = \tilde{w}_0 s_{e_1} \tilde{w}_1 \cdots s_{e_k} \tilde{w}_k$, and there are elements $\tilde{h}_i, h'_i \in G_{e_i}$ for $1 \le i \le k$ such that $w_k =_{G_{\tau(e_k)}} \tilde{h}_k \tilde{w}_k$, $w'_k =_{G_{\tau(e_k)}} h'_k \tilde{w}_k$, and for $k > i \ge 1$, $w_i \phi_{e_{i+1}}(\tilde{h}_{i+1}) =_{G_{\tau(e_i)}} \tilde{h}_i \tilde{w}_i$, $w'_i \phi_{e_{i+1}}(h'_{i+1}) =_{G_{\tau(e_i)}} h'_i \tilde{w}_i$, $w_0 \phi_{e_1}(\tilde{h}_1) =_{G_{v_0}} \tilde{w}_0$, and $w'_0 \phi_{e_1}(\tilde{h}_1) =_{G_{v_0}} \tilde{w}_0$. Hence the elements h_i defined by $h_i h'_i = \tilde{h}_i$ for $1 \le i \le k$ satisfy the properties needed for the conclusion in (ii). $\square$

We note that in the hypotheses of Lemma 3.8(ii), the word w is an arbitrary element of the inflated Higgins language $L' = \widehat{L}(\mathcal{G}, \{(X^{\pm}_{\tau(e)})^*\}, (X^{\pm}_{v_0})^*, v_0, \mathcal{T})$; that is, w is in the inflated Higgins language with respect to the largest possible sets of coset and vertex group representatives. Thus when the $(\mathcal{L}, L_{v_0})$–cascade process is applied to a word in this maximal inflated Higgins language, a word is produced in the inflated Higgins language with respect to more restricted coset and vertex group representatives in $(\mathcal{L}, L_{v_0})$.

Given a word w in a Higgins normal form $w = \mathrm{Defl}(w_0 s_{e_1} u_1 \cdots s_{e_k} u_k, \mathcal{T})$ or in coset normal form $w = \mathrm{Defl}(u_0 s_{e_1} u_1 \cdots s_{e_k} u_k, \mathcal{T})$, the *$\Lambda$–path associated with w* is the directed path $p = e_1 \cdots e_k$ in the graph Λ. An immediate consequence of Lemma 3.8 is that any two words in the inflated Higgins language $\widehat{L}(\mathcal{G}, \mathcal{L}, L_{v_0}, v_0, \mathcal{T})$ that represent the same element of $\pi_1(\mathcal{G})$, or any two words in the inflated Higgins coset language $\widehat{L}(\mathcal{G}, \mathcal{L}, e_0, \mathcal{T})$ that represent the same coset of G_{e_0} in $\pi_1(\mathcal{G})$, have the same associated Λ–path, and so the Λ–path associated with a (deflated) Higgins normal form is well-defined.

Proof of Theorem A. Let $\mathcal{T}$ be a maximal tree in Λ and let $G := \pi_1(\mathcal{G}, \mathcal{T})$. Applying Lemma 2.9, for each $e \in \vec{E}$ we modify the coset language $L^e_{\tau(e)}$ by removing all representatives of the identity coset G_e other than the empty word ϵ. Let

$$X := \cup_{v \in V} X_v \cup \{s_e : e \in \vec{E} \setminus \vec{E}_{\mathcal{T}}\},$$

let $\mathcal{L} := \{L^e_{\tau(e)}\}$ be the collection of (modified) edge coset languages, and let

$$L := L(\mathcal{G}, \mathcal{L}, e_0, \mathcal{T})$$

be the associated Higgins coset language over X. We shall prove that L is the language of a SACA structure for the pair (G, G_{e_0}) over the generating set X.

Now L is a coset language for $(\pi_1(\mathcal{G}), G_{e_0})$ (as discussed in Section 3.1). Let $L_{\tau(e_0)}$ be a regular language of normal forms for $G_{\tau(e_0)}$. It is shown in [8, Prop. 3.3] that the Higgins language for $\pi_1(\mathcal{G})$, with respect to $\mathcal{L}, L_{\tau(e_0)}, \tau(e_0)$, is a regular language; the same proof shows that the Higgins coset language L is also regular.

It now remains to verify fellow traveller properties for L. Let K be a common fellow traveller constant for the coset automatic structures for the pairs $(G_e, G_{\tau(e)})$. Applying Lemmas 2.4 and 2.7 to Hypotheses (ii) and (iii), we can choose $\lambda \in \mathbb{N}$ to be large enough such that:

(a) $K \le \lambda$;

(b) for each $e \in \vec{E}$, the triple $(G_e, G_{\overline{e}}, \phi_e)$ is λ-stable with respect to $(Y_e, Y_{\overline{e}})$;

(c) for each $e, f \in \vec{E}$ with $\tau(e) = \tau(f)$, the coset language $L^e_{\tau(e)}$ has λ-limited crossover with respect to (Y_e, Y_f);

(d) for each $e \in \vec{E}$, any element $g \in G_e$ with $|g|_{X_{\tau(e)}} \le K$ satisfies $|g|_{Y_e} \le \lambda$.

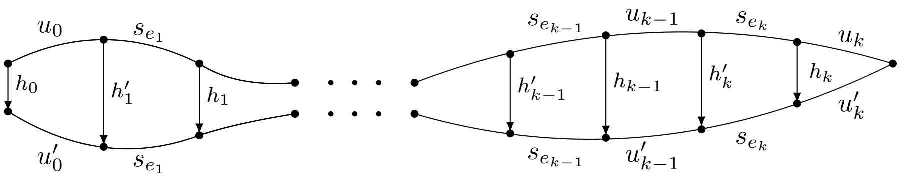

Figure 1: Fellow-travelling words representing the same coset

We define a further constant N to be the maximum value of $|y|_{X_{\tau(e)}}$ for any $e \in \vec{E}$ and any $y \in Y_e$.

Now suppose that $w, w' \in L$ are related by an equation $wx =_G hw'$, where $h \in G_{e_0}$ and $x \in X \cup \{\epsilon\}$; that is, either x is in the generating set of G, or x represents the identity of G.

Let $\widehat{X} := \cup_{v \in V} X_v \cup \{s_e : e \in \vec{E}\}$, let $\widehat{L} := \widehat{L}(\mathcal{G}, \mathcal{L}, e_0, \mathcal{T})$ be the inflated Higgins coset language, and let $L_{e_0} \subseteq (X^{\pm}_{\tau(e_0)})^*$ be a set of words representing the elements of the group G_{e_0}. We suppose that $\widehat{w}, \widehat{w'} \in \widehat{L}$ satisfy $w = \mathrm{Defl}(\widehat{w}, \mathcal{T})$ and $w' = \mathrm{Defl}(\widehat{w'}, \mathcal{T})$, and write $\widehat{w} = u_0 s_{e_1} u_1 \cdots s_{e_k} u_k$. Let $p := e_1 \cdots e_k$ denote the path in Λ determined by w, and let $z_h \in L_{e_0}$ be any word representing the element h in $G_{\tau(e_0)}$.

Case (1): Suppose that $x = \epsilon$. Then $\widehat{w} =_G z_h \widehat{w'}$. The words $\widehat{w}$ and $z_h \widehat{w'}$ are both in the inflated Higgins language $\widehat{L}(\mathcal{G}, \mathcal{L}, L_{e_0} L^{e_0}_{\tau(e_0)}, \tau(e_0), \mathcal{T})$, and so by Lemma 3.8(ii), $z_h \widehat{w'}$ is an $(\mathcal{L}, L_{e_0} L^{e_0}_{\tau(e_0)})$–cascade of $\widehat{w}$ also associated with the path p in Λ. Thus we can write $\widehat{w'} = u'_0 s_{e_1} u'_1 \cdots s_{e_k} u'_k$ with each $u'_i \in L^{e_i}_{\tau(e_i)}$. Moreover, for $1 \le i \le k$ there are elements $h_i \in G_{e_i}$ for which if $h'_i := \phi_{e_i}(h_i) \in G_{\overline{e}_i}$ and $h_0 := h$, then

$$\begin{aligned}
u_k x &=_{G_{\tau(e_k)}} h_k u'_k, & h'_k &=_G s_{e_k} h_k s_{e_k}^{-1},\\
u_{k-1} h'_k &=_{G_{\tau(e_{k-1})}} h_{k-1} u'_{k-1}, & h'_{k-1} &=_G s_{e_{k-1}} h_{k-1} s_{e_{k-1}}^{-1},\\
&\cdots & &\cdots \\
u_1 h'_2 &=_{G_{\tau(e_1)}} h_1 u'_1, & h'_1 &=_G s_{e_1} h_1 s_{e_1}^{-1},\\
u_0 h'_1 &=_{G_{\tau(e_0)}} h_0 u'_0. & &
\end{aligned} \qquad (*)$$

An illustration of the paths ${}_1w$ and ${}_hw'$ in the Cayley graph $\Gamma(G, \widehat{X})$, along with the connector h_i and h'_i paths in this array of equations, is given in Figure 1. We note that in this illustration, for each index j for which $e_j \in \vec{E}_{\mathcal{T}}$, the edges s_{e_j} along the top and bottom paths actually label loops in $\Gamma(G, \widehat{X})$.

We have $|h_k|_{X_{\tau(e_k)}} \le K$, from the fellow traveller property on $L^{e_k}_{\tau(e_k)}$. Our condition (d) ensures that $|h_k|_{Y_{e_k}} \le \lambda$. Then condition (b) ensures that $|h'_k|_{Y_{\overline{e}_k}} \le \lambda$. Then condition (c) ensures that $|h_{k-1}|_{Y_{e_{k-1}}} \le \lambda$. Repeated application of conditions (b) and (c), ensures that, for each i, $|h_i|_{Y_{e_i}} \le \lambda$ and $|h'_i|_{Y_{\overline{e}_i}} \le \lambda$. The definition of the constant N shows that $|h'_i|_{X_{\tau(e_{i-1})}} \le \lambda N$ for each i. Application of the fellow traveller properties for the languages $L^{e_i}_{\tau(e_i)}$ now ensures that ${}_1\widehat{w}$ and ${}_h\widehat{w'}$ asynchronously fellow travel at distance $KN\lambda$ in $\Gamma(G, \widehat{X})$.

The paths ${}_1w$ and ${}_hw'$ in the Cayley graph $\Gamma(G, X)$ are obtained from the paths ${}_1\widehat{w}$ and ${}_h\widehat{w'}$ by skipping the s_{e_j} edges in both paths whenever $e_j \in \vec{E}_{\mathcal{T}}$.

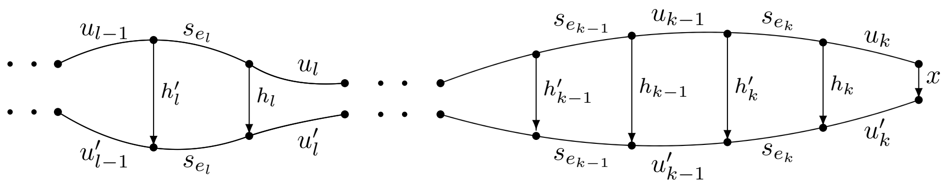

Figure 2: Fellow-travelling words, case (2)

Thus the paths ${}_1w$ and ${}_hw'$ also asynchronously fellow travel at distance $KN\lambda$ in $\Gamma(G,X)$.

Case (2): Suppose that $x \in X_{\tau(e_k)}$. Let u be an $(\mathcal{L}, L_{e_0}L^{e_0}_{\tau(e_0)})$–cascade of wx. Then the Λ–path associated with u is a prefix $p' = e_1 \cdots e_\ell$ of p. Since the word $z_h\widehat{w'}$ is in the inflated Higgins language $\widehat{L}(\mathcal{G}, \mathcal{L}, L_{e_0}L^{e_0}_{\tau(e_0)}, \tau(e_0), \mathcal{T})$ and satisfies $u =_G z_h\widehat{w'}$, it follows from by Lemma 3.8 that the word $z_h\widehat{w'}$ is an $(\mathcal{L}, L_{e_0}L^{e_0}_{\tau(e_0)})$–cascade of u, and the path in Λ associated with $z_h\widehat{w'}$ is also p'. So $\widehat{w'} = u'_0 s_{e_1} u'_1 \cdots s_{e_\ell} u'_\ell$ with $\ell \le k$ and each $u'_i \in L^{e_i}_{\tau(e_i)}$. In the case that $\ell < k$, let $u'_{\ell+1} = \cdots = u'_k = \epsilon$.

Now a composition of two cascades is again a cascade, and so $z_h\widehat{w'}$ is an $(\mathcal{L}, L_{e_0}L^{e_0}_{\tau(e_0)})$–cascade of the word wx. Hence there are elements $h_i \in G_{e_i}$ for $1 \le i \le k$ for which if $h'_i := \phi_{e_i}(h_i) \in G_{\overline{e}_i}$ and $h_0 := h$, then the array in Equation $(*)$ holds. The corresponding paths in the Cayley graph $\Gamma(G, \widehat{X})$ are illustrated in Figure 2.

Then just as in Case (1) we can bound the lengths of each h_i, h'_i over appropriate generating sets by λ, and see that ${}_1w$ and ${}_hw'$ asynchronously $KN\lambda$–fellow travel in $\Gamma(G,X)$.

Case (3): Suppose that $x \in X_v$, for some vertex v, but $x \notin X_{\tau(e_k)}$. In this case we extend the path p in Λ that corresponds to w to a path p' by appending the unique minimal path $e_{k+1} \cdots e_\ell$ within the tree $\mathcal{T}$ from $\tau(e_k)$ to v; then $\tau(e_\ell) = v$. We consider the word $\widetilde{w} := \widehat{w} s_{e_{k+1}} u_{k+1} \cdots s_{e_\ell} u_\ell x$, with $u_{k+1} = \cdots = u_\ell = \epsilon$. Since $\widehat{w}$ does not end with a letter s_e for an edge e in $\mathcal{T}$, the word $\widetilde{w}$ satisfies the hypotheses of the word w in Lemma 3.8(i).

Let u be an $(\mathcal{L}, L_{e_0}L^{e_0}_{\tau(e_0)})$–cascade of $\widetilde{w}$; by applying both parts of Lemma 3.8, we see that the word $z_h\widehat{w'}$ is an $(\mathcal{L}, L_{e_0}L^{e_0}_{\tau(e_0)})$–cascade of u, and hence also of $\widetilde{w}$. Now the Λ–path associated with both u and $z_h\widehat{w'}$ may be a prefix $e_1 \cdots e_j$ of p'; we can write $\widehat{w'} = u'_0 s_{e_1} u'_1 \cdots s_{e_j} u'_j$ with $j \le \ell$ and each $u'_i \in L^{e_i}_{\tau(e_i)}$, and in the case that $j < \ell$, let $u'_{j+1} = \cdots = u'_\ell = \epsilon$. The cascade from $\widetilde{w}$ to $z_h\widehat{w'}$ now yields elements $h_i \in G_{e_i}$ for $1 \le i \le \ell$, which together with the elements $h'_i := \phi_{e_i}(h_i) \in G_{\overline{e}_i}$ and $h_0 := h$ satisfy the array in Equation (*). The corresponding paths in the Cayley graph $\Gamma(G, \widehat{X})$ are illustrated in Figure 3.

Then, as in Case (1), we deduce that ${}_1w$ and ${}_hw'$ asynchronously fellow travel in $\Gamma(G,X)$ at a distance bounded by $KN\lambda$.

Case (4): Suppose that $x = s_e$ for some $e \in \vec{E} \setminus \vec{E}_{\mathcal{T}}$. If $u_k \ne \epsilon$ or $e \ne \overline{e}_k$, then the word $\widehat{w} s_{e_{k+1}} \cdots s_{e_\ell} x$ is in the inflated Higgins coset language $\widehat{L}$, where $e_{k+1} \cdots e_\ell$ is the unique minimal path (possibly empty) in the tree $\mathcal{T}$ from $\tau(e_k)$ to the initial vertex of e. Then $wx = \mathrm{Defl}(\widehat{w} s_{e_{k+1}} \cdots s_{e_\ell} x, \mathcal{T})$ is in the Higgins coset language L. In this subcase the proof in Case (1) shows that the paths

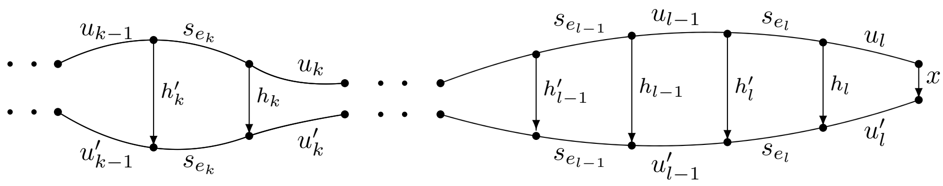

Figure 3: Fellow-travelling words, case (3)

${}_1wx$ and ${}_hw'$ in $\Gamma(G,X)$ $KN\lambda$–fellow travel.

So now suppose that $u_k = \epsilon$ and $e = \overline{e}_k$. In that case we can write $\widehat{w} = \widetilde{w}s_{e_k}$ with $\widetilde{w} \in (\widehat{X}^{\pm})^*$, and also write $\widetilde{w} = \widetilde{w}''s_{e_j}\cdots s_{e_{k-1}}$, where $s_{e_j}\cdots s_{e_{k-1}}$ is the maximal suffix of $\widetilde{w}$ lying in $\{s_e \mid e \in \vec{E}_{\mathcal{T}}\}^*$. That is, $\widetilde{w}''$ is obtained from $\widehat{w}$ by removing the letter s_{e_k} at the end, and then removing any resulting suffix of generators s_{e_j} associated with edges lying in tree $\mathcal{T}$. Now $\widetilde{w}''$ is in the inflated Higgins coset language $\widehat{L}$, and so the word $w'' := \mathrm{Defl}(\widetilde{w}'',\mathcal{T})$ is in the Higgins coset language L. Moreover, $w'' =_G wx =_G hw'$, and so Case (1) applies to show that the paths ${}_1w''$ and ${}_hw'$ in $\Gamma(G,X)$ asynchronously $KN\lambda$–fellow travel. Since $w = w''s_{e_k}$, the paths ${}_1w$ and ${}_hw'$ asynchronously $KN\lambda+1$–fellow travel in $\Gamma(G,X)$. □

3.4 Finding synchronous subsystems

Note that we might expect that an argument analogous to the proof of Theorem A would allow us to derive a synchronous fellow traveller property for L from synchronous fellow traveller properties for the coset languages $L^e_{\tau(e)}$. However it is not clear that this is possible, since it seems likely that for words $\mathrm{Defl}(u_0s_{e_1}u_1\cdots s_{e_k}u_k,\mathcal{T})$ and $\mathrm{Defl}(u'_0s_{e'_1}u'_1\cdots s_{e'_{k'}}u'_{k'},\mathcal{T})$ as above, representing the same coset, the lengths of the corresponding subwords u_j and u'_j could differ.

But, as we prove in Proposition 3.9 below, under certain conditions a strong asynchronous automatic coset system must contain a synchronous system as a substructure. We shall use this result to derive Theorem B and other synchronous results relating to Theorem A. Our proof of the proposition emulates the proof of [15, Lemma 1], which shows that two geodesic paths that start at 1 in a Cayley graph and and asynchronously K-fellow travel must also synchronously $2K$-fellow travel.

Proposition 3.9. *Suppose that (G,H) has a strong asynchronous automatic coset system L^H for which $L^H(\textsc{Geo}) := L^H \cap \textsc{Geo}^H$ is a coset language (contains at least one representative of each coset). Then $L^H(\textsc{Geo})$ is a strong synchronous automatic coset system for (G,H).*

Proof. We first prove regularity of $L^H(\textsc{Geo})$ by proving regularity of its complement in L^H. Let X be the generating set for G and let K be the asynchronous fellow traveller constant associated with $L^H \subseteq (X^{\pm})^*$, and let N be the number of states in the automaton recognising L^H.

Suppose that $w \in L^H \setminus L^H(\textsc{Geo})$, and let w' be the shortest prefix of w that is not of minimal length within its coset. Then there exists $v \in L^H(\textsc{Geo})$ with $|v| < |w'|$ and $v \in Hw'$, and there exists a word $w'u \in L^H$, with $|u| < N$. Let $h \in H$ with $w' =_G hv$. Then, since $v, w'u \in L^H$ with $(Hv)u =_G Hw'u$,

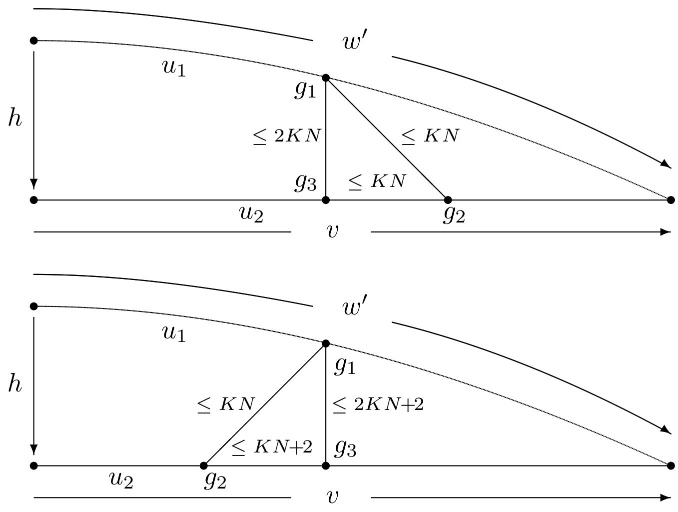

Figure 4: ${}_1w'$ and ${}_hv$ synchronously fellow travel

the fellow traveller condition on L^H implies that the paths ${}_1w'u$ and ${}_hv$ in $\Gamma(G,X)$ asynchronously KN-fellow travel. Note that this implies in particular that $|h|_X \leq KN$.

We shall now show that ${}_1w'$ and ${}_hv$ synchronously fellow travel with constant $2KN+2$. Take any vertex g_1 of $\Gamma(G)$ on the path ${}_1w'$, and let g_2 be a vertex of $\Gamma(G)$ on the path ${}_hv$ that is closest to g_1. Let u_1 be the prefix of w' labeling the subpath of ${}_1w'$ from 1 to g_1, and u_2 the label of the subpath of ${}_hv$ from h to g_2.

Now, both v and the maximal proper prefix of w' are shortest representatives of their cosets of H, and any prefix of a word in Geo^H is also in Geo^H. Then $u_2 \in \text{Geo}^H$, and either $u_1 = w'$ or $u_1 \in \text{Geo}^H$. Then we have $|u_2| \leq |u_1| + KN$ and $|u_1| \leq |u_2| + KN + 2$, and hence $||u_1| - |u_2|| \leq KN + 2$. So now the vertex g_3 of ${}_hv$ that is at distance $|u_1|$ from h is at distance at most $KN+2$ from g_2, and hence distance at most $(KN+2) + KN = 2KN+2$ from g_1 (see Fig. 4). This verifies our claim that ${}_1w'$ and ${}_hv$ synchronously $(2KN+2)$-fellow travel.

Using the elements of G in the ball of radius $2KN+2$ centred at 1 (or "word differences") in constructing a finite set of states, we can construct a finite state automata to recognise the languages (of padded pairs)

$$\begin{aligned} L_h \quad := \quad & \{(w,v) \mid w,v \in L^H \text{ and } w \text{ has a prefix } w' \text{ with } w' =_G hv, |w'| > |v|, \\ & \text{and } {}_1w', {}_hv \text{ synchronously } (2KN+2)\text{–fellow travel}\} \end{aligned}$$

for each $h \in H$ with $|h|_X \leq KN$. (See [16] or [20] for more details on word difference machines and padding to convert a language of pairs of words to a language of words over a product alphabet.) The language $L^H \setminus L^H(\text{Geo})$ is the union of the projections onto the first coordinate of the sets L_h. Since regularity is preserved by projection, we see that the complement of $L^H(\text{Geo})$ is indeed regular, and hence so is $L^H(\text{Geo})$.

Now suppose that $v, w \in L^H(\text{Geo})$ and $h \in H$, with $d_{\Gamma(G)}(w, hv) \leq 1$. Then, much as above, we see that ${}_1w$ and ${}_hv$ synchronously fellow travel with constant $2K$. For now, if g_1 is any vertex of $\Gamma(G)$ on the path ${}_1w$, g_2 a vertex of $\Gamma(G)$

that is closest to g_1 on the path ${}_hv$, u_1 the label of the subpath of ${}_1w'$ from 1 to g_1, u_2 the label of the subpath of ${}_hv$ from h to g_2, and g_3 the vertex of ${}_hv$ at distance $|u_1|$ from h, then $|u_2| \leq |u_1| + K$ and $|u_1| \leq |u_2| + K$, and hence $||u_1| - |u_2|| \leq K$, and $d_{\Gamma(G)}(g_1, g_3) \leq 2K$. □

3.5 A synchronous result for graphs of groups

Theorem B. *Let $\mathcal{G} = (\Lambda = (V, \vec{E}), \{G_v : v \in V\}, \{G_e : e \in \vec{E}\}, \{\phi_e \mid e \in \vec{E}\})$ be a graph of groups over a finite connected directed graph Λ with an edge e_0. Let X_v and Y_e be finite generating sets of the groups G_v and G_e, respectively. Suppose that the following conditions hold for each $e \in \vec{E}$.*

- *(i)* $Y_e \subseteq X_{\tau(e)}$.
- *(ii)* *The pair $(G_{\tau(e)}, G_e)$ is strongly synchronously coset automatic with coset language $L^e_{\tau(e)}$ satisfying $L^e_{\tau(e)} \subset \mathrm{GEO}(G_{\tau(e)}, X_{\tau(e)}) \cap [(X^{\pm}_{\tau(e)})^* \backslash Y^{\pm}_e (X^{\pm}_{\tau(e)})^*]$, the only representative in $L^e_{\tau(e)}$ of the identity coset is ϵ, and each element $g \in G_{\tau(e)}$ is represented by a word $y_g z_g \in \mathrm{GEO}(G_{\tau(e)}, X_{\tau(e)})$ with $y_g \in (Y^{\pm}_e)^*$ and $z_g \in L^e_{\tau(e)}$.*
- *(iii)* *The triple $(G_e, G_{\overline{e}}, \phi_e)$ is 1–stable with respect to $(Y_e, Y_{\overline{e}})$.*
- *(iv)* *For each $f \in \vec{E}$ with $\tau(e) = \tau(f)$, the coset language $L^e_{\tau(e)}$ has limited crossover with respect to (Y_e, Y_f).*

Then the pair $(\pi_1(\mathcal{G}), G_{e_0})$ is strongly synchronously coset automatic.

Proof. Let $G := \pi_1(\mathcal{G})$, $H := G_{e_0}$, and $X := \cup_{v \in V} X_v$, and let $\mathcal{T}$ be any tree in Λ. Let $\mathcal{L} := \{L^e_{\tau(e)} \mid e \in \vec{E}\}$. Then Theorem A shows that the pair $(G, H) = (\pi_1(\mathcal{G}), G_{e_0})$ is SACA, with respect to the Higgins coset language $L^H := L(\mathcal{G}, \mathcal{L}, e_0, \mathcal{T}) \subseteq (X^{\pm})^*$.

By Proposition 3.9, it suffices to show that $L^H \cap \mathrm{GEO}^H_G(X)$ contains at least one representative of each coset. Note that the empty word is in $L^H \cap \mathrm{GEO}^H_G(X)$.

Let w be any nonempty element of $\mathrm{GEO}^H_G(X)$; that is, w is of minimal length as a representative over X of the right coset Hw in G. Write $w = x_1 \cdots x_m$ with each $x_i \in X^{\pm}$. For each index i such that $x_i = s_e^{-1} \in \{s_{e'}^{-1} \mid e' \in \vec{E} \backslash \vec{E}_{\mathcal{T}}\}$, replace x_i by $s_{\overline{e}}$, to obtain a word $w' = \tilde{x}_1 \cdots \tilde{x}_m$ over $(\cup_{v \in V} X^{\pm}_v) \cup \{s_{e'} \mid e' \in \vec{E} \setminus \vec{E}_{\mathcal{T}}\}$. For $1 \leq i \leq m$, if $\tilde{x}_i \in \cup_{v \in V} X^{\pm}_v$ then let $v_i = v'_i$ be the unique vertex in Λ for which $\tilde{x}_i \in X_{v_i}$ and if $\tilde{x}_i \in \{s_{e'} \mid e' \in \vec{E} \setminus \vec{E}_{\mathcal{T}}\}$ then let v_i and v'_i be the initial and terminal vertices, respectively, of the edge e for which $\tilde{x}_i = s_e$. Let $t_0 \in \{s_e \mid e \in \vec{E}_{\mathcal{T}}\}^*$ be the (possibly empty) word corresponding to the geodesic path in the tree $\mathcal{T}$ from $\tau(e_0)$ to the vertex v_1. Similarly for $1 \leq i \leq m-1$ let $t_i \in \{s_e \mid e \in \vec{E}_{\mathcal{T}}\}^*$ be the word corresponding to the geodesic in $\mathcal{T}$ from v'_i to v_{i+1}. Let $\widehat{w} := t_0 \tilde{x}_1 t_1 \cdots t_{m-1} \tilde{x}_m$. Then $w' = \mathrm{Defl}(\widehat{w}, \mathcal{T})$.

Repartitioning the subwords of $\widehat{w}$, we can write $\widehat{w} = u_0 s_{e_1} u_1 \cdots s_{e_k} u_k$ with $u_i \in (X^{\pm}_{\tau(e_i)})^*$ for each i, and $e_1 \cdots e_k$ is a path in Λ starting at $\tau(e_0)$. Since the original word w is a geodesic over X, and each u_i is a subword of w, we have $u_i \in \mathrm{GEO}(G_{\tau(e_i)}, X_{\tau(e_i)})$ for all i.

Next we construct a choice of $(\mathcal{L}, (Y^{\pm}_{e_0})^* L^{e_0}_{\tau(e_0)})$–cascade of $\widehat{w}$. By hypothesis (ii), the element of $G_{\tau(e_k)}$ represented by u_k is also represented by a word of the form $y_k u'_k \in \mathrm{GEO}(G_{\tau(e_k)}, X_{\tau(e_k)})$ with $y_k \in (Y^{\pm}_{e_k})^*$ and $u'_k \in L^{e_k}_{\tau(e_k)}$. Note that if u_k represents an element of G_{e_k}, then $u'_k = \epsilon$. Then

$|y_k| + |u'_k| = |u_k|$. Now the 1–stability condition says that there is a word $y'_k \in (Y^{\pm}_{e_{k-1}})^*$ with $y'_k =_{G_{e_k}} \phi_{e_k}(y_k) =_G s_{e_k} y_k s_{e_k}^{-1}$ and $|y'_k| = |y_k|$. Next there is word $y_{k-1}u'_{k-1} \in \mathrm{G{\small EO}}(G_{\tau(e_{k-1})}, X_{\tau(e_{k-1})})$ representing the element $u_{k-1}y'_k$ of $G_{\tau(e_{k-1})}$, with $y_{k-1} \in (Y^{\pm}_{e_{k-1}})^*$ and $u'_{k-1} \in L^{e_{k-1}}_{\tau(e_{k-1})}$ (and again $u'_{k-1} = \epsilon$ if $u_{k-1}y'_k$, and hence also u_{k-1}, represents an element of $G_{e_{k-1}}$). Then $|y_{k-1}| + |u'_{k-1}| \le |u_{k-1}| + |y'_k|$. Repeating this process, we obtain the word $\widehat{w}' = y_0 u'_0 s_{e_1} u'_1 \cdots s_{e_k} u'_k$ satisfying

$$\begin{aligned} |y_0| + \sum_{j=0}^{k} |u'_j| &\le |u_0| + |y'_1| + \sum_{j=1}^{k} |u'_j| \le \cdots \le \left(\sum_{j=0}^{i-1} |u_j|\right) + |y'_i| + \left(\sum_{j=i}^{k} |u'_j|\right) \\ &\le \cdots \le \sum_{j=0}^{k} |u_j|. \end{aligned}$$

Let $\widehat{w}'' = u''_0 s_{e''_1} u''_1 \cdots s_{e''_\ell} u''_\ell$ be the word obtained from $\widehat{w}'$ by removing the y_0 prefix, removing the maximal suffix in $\{s_e \mid e \in \vec{E}_{\mathcal{T}}\}^*$, and (iteratively) removing any subwords of the form $s_e s_{\overline{e}}$ with $e \in \vec{E}_{\mathcal{T}}$.

Let $w'' := \mathrm{Defl}(\widehat{w}'', \mathcal{T})$. Then w'' represents the same coset of H in G as the word $w \in \mathrm{G{\small EO}}^H_G(X)$. Since the words w, w'' contain the same number of letters in $\{s_e \mid e \in \vec{E} \setminus \vec{E}_{\mathcal{T}}\}$ (because the inflation, cascade, and deflation processes don't alter those letters), we have $\sum_{j=0}^{k} |u_j| \le \sum_{j=0}^{\ell} |u''_j| = \sum_{j=0}^{k} |u'_j|$. Hence these sums are equal, $y_0 = \epsilon$, and $w'' \in \mathrm{G{\small EO}}^H_G(X)$ as well.

The fact that ϵ is the only representative of the identity coset G_e in each $L^e_{\tau(e)}$ guarantees that either $\ell = 0$ or $e_\ell \notin \vec{E}_{\mathcal{T}}$ or u''_ℓ does not represent an element of G_{e_ℓ}, and similarly guarantees that, for each subword $s_{e''_i} u''_i s_{e''_{i+1}}$ of $\widehat{w}''$, either $u''_i = \epsilon$ or u''_i does not represent an element of G_{e_i}. By construction, $\widehat{w}''$ doesn't contain a subword of the form $s_e s_{\overline{e}}$ for any $e \in \vec{E}_{\mathcal{T}}$. If the word $\widehat{w}''$ contains a subword of the form $s_e s_{\overline{e}}$ with $e \in \vec{E} \setminus \vec{E}_{\mathcal{T}}$, then so does the deflated word w'', contradicting the fact that w'' is a geodesic word over X. Then $\widehat{w}''$ is in the inflated Higgins coset language $\widehat{L}(\mathcal{G}, \mathcal{L}, e_0, \mathcal{T})$, and so its deflation w'' is in the language L^H.

Therefore w'' is an element of $L^H \cap \mathrm{G{\small EO}}^H_G(X)$ representing the same coset as the original word w. Hence $L^H \cap \mathrm{G{\small EO}}^H_G(X)$ is a coset language for H in G, as required. □

4 Automaticity for graphs of relatively hyperbolic groups

This section provides examples of groups with strong synchronous automatic coset systems that satisfy the crossover conditions that we would need in order to apply Theorem A, specifically certain relatively hyperbolic groups. We begin in Section 4.1 with some background and an account of relevant existing results for relatively hyperbolic groups. In Section 4.2 we discuss crossover and SSCA for relatively hyperbolic groups. Finally, in Section 4.3 we prove (in Corollary 4.10) automaticity for any fundamental group of a graph of groups in which the vertex groups are hyperbolic relative to abelian groups, the edge groups are peripheral subgroups of the vertex groups, and a further hypothesis holds on paths in the graph. Then in Corollary C we give an application to 3-manifold groups.

4.1 Background on relatively hyperbolic groups and biautomaticity

Background and details on relatively hyperbolic groups and biautomatic structures used in this paper can be found in [3, 25, 16].

Let $G = \langle X \rangle$ be a group with finite generating set X. For any path p in $\Gamma(G, X)$, let $\iota(p)$ denote the initial vertex, and let $\tau(p)$ denote the terminal vertex, of p. Given $\lambda \geq 1$ and $c \geq 0$, the path p is a (λ, c)–*quasigeodesic* if for every subpath r of p, the inequality $l(r) \leq \lambda d_{\Gamma(G,X)}(\iota(r), \tau(r)) + c$ holds.

The group G is *biautomatic* if there is a regular language L for G (over X) and a constant $K \geq 0$ satisfying the property that whenever $u, v \in L$ and $x, y \in X^{\pm} \cup \{\epsilon\}$ satisfy $ux =_G yv$, the paths ${}_1y^{-1}ux$ and ${}_1v$ synchronously K–fellow travel [16, Lemma 2.5.5].

Let $\{H_\omega \mid \omega \in \Omega\}$ be a collection of subgroups of G, and let $\mathcal{H} = \cup_{\omega \in \Omega}(H_\omega \setminus \{1\})$. The graph $\Gamma(G, X \cup \mathcal{H})$ is called the *relative Cayley graph* of G.

Given a path p in $\Gamma(G, X \cup \mathcal{H})$, the path p *penetrates* the coset gH_ω if p contains an edge labelled by an element of H_ω that connects two vertices of gH_ω. An H_ω–*component* of such a path is a non-empty maximal subpath of p that is labelled by a word in H_ω^*. The path p is said to be *without backtracking* if, whenever $p = p'srs'p''$ with two H_ω–components s, s', the initial vertices of s and s' lie in different left cosets of H_ω (intuitively, p penetrates every left coset at most once). The path p is *without vertex backtracking* if each subpath of p of length at least 2 is labelled by a word that does not represent an element of an H_ω subgroup. In particular, if a path does not vertex backtrack, then it does not backtrack and all components are edges.

Following [25] we say that G is *hyperbolic relative to* $\{H_\omega\}$ if the following two conditions hold.

(i) $\Gamma(G, X \cup \mathcal{H})$ is Gromov-hyperbolic.

(ii) Given any $\lambda \geq 1$, there is a constant $B(\lambda)$ with the following property. Let p and q be any two $(\lambda, 0)$–quasigeodesic paths without backtracking in $\Gamma(G, X \cup \mathcal{H})$ with $\iota(p) = \iota(q) = 1$ and $d_{\Gamma(G,X)}(\tau(p), \tau(q)) \leq 1$. Then:

   (a) if s is an H_ω–component of p penetrating the coset gH_ω, and q does not penetrate gH_ω, then the distance between the initial and terminal vertices of s in $\Gamma(G, X)$ is at most $B(\lambda)$;

   (b) if s is an H_ω–component of p penetrating the coset gH_ω and s' is an H_ω–component of q penetrating the same coset, then in $\Gamma(G, X)$ the distance between the initial vertices of s and s', and the distance between the terminal vertices of s and s', are both at most $B(\lambda)$.

Property (i) is frequently called *weak relative hyperbolicity* and Property (ii) is frequently called *bounded coset penetration*. The groups H_ω are called the *peripheral subgroups* of the hyperbolic group G.

Remark 4.1. In a finitely generated relatively hyperbolic group G, bounded coset penetration also holds for (λ, c)–quasigeodesics with $\lambda \geq 1$ and $c \geq 0$, with a constant $B(\lambda, c)$ [25, Theorem 3.23].

Relatively hyperbolic groups with a finite generating set satisfy several further fundamental properties that we shall use.

Lemma 4.2. *Let G be a finitely generated group hyperbolic relative to a collection $\{H_\omega\}$ of subgroups. Then*

(i) [25, Corollary 2.48] there are only finitely many groups H_ω; that is, $|\Omega| < \infty$;
(ii) [25, Proposition 2.36] for all $\omega, \mu \in \Omega$ with $\omega \neq \mu$, the intersection $H_\omega \cap H_\mu$ is finite;
(ii) [25, Proposition 2.29] each H_ω is finitely generated.

Definition 4.3. [3, Construction 4.1] Let w be a word in $(X^\pm)^*$; we define the *factorisation* of w to be its decomposition as $w = w_0 u_1 w_1 \cdots u_n w_n$ where

(i) each w_k is in $(X^\pm \setminus (\cup_{\omega\in\Omega}(X^\pm \cap H_\omega)))^*$,
(ii) each u_k is a nonempty word in $(X^\pm \cap H_{\omega_k})^*$ for some $\omega_k \in \Omega$,
(iii) if $w_k = \epsilon$ and x is the first letter of u_{k+1}, then $u_k x$ is not in $(X^\pm \cap H_\omega)^*$ for any ω.

We define the *derived word* $\hat{w}$ of w to be the word $\hat{w} := w_0 h_1 w_1 \cdots h_n w_n$ over $X^\pm \cup \mathcal{H}$, where each h_k is the element of $\mathcal{H}$ represented by u_k (or $h_k = \epsilon$ if $u_k =_G 1$). Similarly, if p is a path in $\Gamma(G, X)$ labelled by w, then the *derived path* $\hat{p}$ is the corresponding path in $\Gamma(G, X \cup \mathcal{H})$ labelled by $\hat{w}$.

Following the notation in [3, Definition 4.5], given subsets $L_{H_\omega} \subseteq (X^\pm \cap H_\omega)^*$ for each $\omega \in \Omega$, let $\mathsf{Rel}(X, \{L_\omega\}^{\mathsf{pc}})$ denote the set of all words w in $(X^\pm)^*$ such that, in the factorisation $w = w_0 u_1 w_1 \cdots u_n w_n$ of w, each u_k is a prefix of a word in $\cup_{\omega\in\Omega} L_\omega$.

The following result, which we state here as a lemma, is a combination of several results in [3]. We use it to prove Proposition 4.5.

Lemma 4.4. *Let $G = \langle X_1 \rangle$ (with $|X_1| < \infty$) be hyperbolic relative to a family of subgroups $\{H_\omega\}_{\omega\in\Omega}$. Then there exist constants $\lambda \geq 1$ and $c \geq 0$ and a finite subset $\mathcal{H}'$ of $\mathcal{H} = \cup_{\omega\in\Omega}(H_\omega \setminus \{1\})$ such that, whenever X is a finite set satisfying*

(i) $X_1 \cup \mathcal{H}' \subseteq X \subseteq X_1 \cup \mathcal{H}$, and
(ii) for all $\omega \in \Omega$, the group H_ω has a geodesic biautomatic structure over $H_\omega \cap X$ with language L_{H_ω},

the following hold.

(a) For every $\omega, \mu \in \Omega$ with $\omega \neq \mu$, the intersection $H_\omega \cap H_\mu$ is contained in X.
(b) For every $\omega \in \Omega$, the set $X \cap H_\omega$ generates H_ω.
(c) For any word $w \in \textsc{Geo}(G, X)$, the word $\hat{w} \in (X^\pm \cup \mathcal{H})^*$ derived from w labels a (λ, c)–quasigeodesic path in $\Gamma(G, X \cup \mathcal{H})$ without vertex backtracking.

(d) *For every $\omega \in \Omega$, if $w \in \textsc{Geo}(G,X)$ represents an element of H_ω, then $w \in \textsc{Geo}(H_\omega, X \cap H_\omega)$.*

(e) *The group G has a biautomatic structure over X with language $\textsc{Geo}(G,X) \cap \mathsf{Rel}(X, \{L_{H_\omega}\}^{\mathsf{pc}})$.*

Proof. By Lemma 4.2, there are finitely many peripheral subgroups, and they have pairwise finite intersections; hence the subset $\mathcal{H}_1 := \cup_{\omega \neq \mu \in \Omega}(H_\omega \cap H_\mu \setminus \{1\})$ of $\mathcal{H}$ is finite. It also follows from Lemma 4.2 that each peripheral subgroup H_ω has a finite generating set Y_ω, and so the subset $\mathcal{H}_2 := \cup_{\omega \in \Omega} Y_\omega$ of $\mathcal{H}$ is finite. Let $\mathcal{H}_3$ be the finite subset $\mathcal{H}'$ of [3, Lemma 5.3], and let $\mathcal{H}_4$ be the finite subset $\mathcal{H}'$ of [3, Theorem 7.6]. Then the finite subset $\mathcal{H}' := \cup_{i=1}^{4} \mathcal{H}_i$ satisfies (a)–(b), and (c) and (e) follow from the two results of Antolin and Ciobanu. Suppose that $w \in \textsc{Geo}(G,X)$ represents an element of H_ω, and let $w = w_0 u_1 w_1 \cdots u_n w_n$ be the factorisation of w. Since, by (c), the word $\hat{w}$ derived from w has no vertex backtracking, the word $\hat{w}$ must have length at most 1; hence $w \in (X^{\pm} \cap H_\omega)^*$, proving (d). □

4.2 Crossover properties for relatively hyperbolic groups

In this section we use Lemma 4.4 to show that a group that is hyperbolic relative to geodesically biautomatic subgroups is coset automatic relative to each peripheral subgroup with maximal crossover. We note that a similar but weaker SSCA result is shown in [8, Theorem 5.4].

Proposition 4.5. *Let $G = \langle X_1 \rangle$ (with $|X_1| < \infty$) be a group that is hyperbolic relative to subgroups $\{H_\omega \mid \omega \in \Omega\}$. Suppose that, for each ω, any finite generating set for H_ω can be extended to one over which H_ω has a geodesic biautomatic structure. Let $\omega_0 \in \Omega$, and let $H := H_{\omega_0}$. Then there are constants $\lambda \geq 1$ and $c \geq 0$ and a finite generating set X for G satisfying the following.*

(1) *The set X satisfies properties (a)–(e) of Lemma 4.4, and hence the subgroup H is generated by $Y := X \cap H_{\omega_0}$.*

(2) *The pair (G,H) is strongly synchronously coset automatic with respect to a coset language L^H satisfying $L^H \subset \textsc{Geo}(G,X) \cap [(X^{\pm})^* \setminus Y^{\pm}(X^{\pm})^*]$, the only representative in L^H of the identity coset is ϵ, and each element $g \in G$ is represented by a word $y_g z_g \in \textsc{Geo}(G,X)$ with $y_g \in (Y^{\pm})^*$ and $z_g \in L^H$.*

(3) *For all $\omega \in \Omega$ the language L^H has maximal crossover with respect to $(Y, X \cap H_\omega)$.*

We note that the condition on finite generating sets of the H_ω holds when each subgroup H_ω is either virtually abelian (by [3, Prop 10.1]) or hyperbolic.

Proof. Given the finite generating set X_1 of G, let $\lambda \geq 1$ and $c \geq 0$ be the constants and let $\mathcal{H}'$ be the subset of $\mathcal{H}$ from (the proof of) Lemma 4.4. Let $X_2 := X_1 \cup \mathcal{H}'$. For each $\omega \in \Omega$, the set $X_2 \cap H_\omega$ generates H_ω (by Lemma 4.4(b)), and so there is another finite generating set $Y_\omega \supseteq X_2 \cap H_\omega$ over which H_w has a biautomatic structure, with a language L_{H_ω}. Let $X := X_2 \cup (\cup_{\omega \in \Omega} Y_\omega)$. Then $X_1 \cup \mathcal{H}' \subseteq X \subset X_1 \cup \mathcal{H}$. Moreover, since $H_\omega \cap H_\mu \subseteq \mathcal{H}' \subseteq X_2$ for all $\omega \neq \mu$, we have $X \cap H_\omega = Y_\omega$ for all ω. Now X is a finite generating set for G satisfying (i)–(ii) of Lemma 4.4, and so properties (a)–(e) of that lemma hold, which proves (1).

Let $H := H_{\omega_0}$ and $Y := X \cap H_{\omega_0}$. Let

$$L := \mathrm{Geo}(G, X) \cap \mathsf{Rel}(X, \{L_{H_\omega}\}^{\mathsf{pc}}),$$

be the language of the biautomatic structure for G over X (from Lemma 4.4(e)). Finally, let

$$L^H := L \cap [(X^\pm)^* \setminus Y^\pm (X^\pm)^*];$$

that is, L^H is the set of words in the geodesic biautomatic structure for G that do not begin with a letter in $Y^\pm$. Since L is regular, and the class of regular languages is closed under intersection, complementation, and concatenation, the language L^H is also regular.

For any element $g \in G$, there is a word $w \in L \subseteq \mathrm{Geo}(G, X)$ representing g, and we can write $w = y_g z_g$ where y_g is the maximal prefix of w lying in $(Y^\pm)^*$ and z_g does not start with a letter in $Y^\pm$. The factorisation of w is y_g followed by the factorisation of z_g; in particular, the suffix z_g of w is also a geodesic over X for which the components lie in the prefix closures of the geodesic biautomatic structures of the component subgroups, and so $z_g \in L^H$. Moreover, z_g is a representative in L^H of the coset Hg. Thus L^H is a coset language for (G, H).

Let $w \in L^H$ be a representative of the identity coset; that is, $w \in H$. Then it follows from Lemma 4.4(d) that $w \in \mathrm{Geo}(H, Y) \subset (Y^\pm)^*$, but no word in L^H begins with a letter in $Y^\pm$. Thus $w = \epsilon$ in this case.

Before proving that the language L_H satisfies the requisite fellow traveller and crossover properties, we prove two lemmas.

Lemma 4.6. *Let $v \in L^H$ and let $\hat{v}$ be the derived word defined in Definition 4.3. Then any path in $\Gamma(X \cap \mathcal{H})$ labelled by $\hat{v}$ is a (λ, c)–quasigeodesic that does not vertex backtrack, and no such path of the form ${}_h\hat{v}$ with $h \in H$ penetrates the coset H.*

Proof. Since $L^H \subseteq \mathrm{Geo}(G, X)$, the first claim follows immediately from Lemma 4.4(c). For any $h \in H$, if the path $p := {}_h\hat{v}$ were to penetrate the identity coset H, then we could write $p = rst$, where s is an edge labelled by a letter in H, and the initial and terminal vertices of s lie in H. However, since (by the definition of L^H) the first letter of v cannot lie in H, the path r is nonempty, and so rs is a path of length at least 2 labelled by a word representing an element of H, contradicting the fact that p has no vertex backtracking. So p cannot penetrate the coset H. □

Lemma 4.7. *Suppose that the word $y \in \mathrm{Geo}(G, X)$ represents an element of $H = H_{\omega_0}$ that does not lie in H_ω for any $\omega \neq \omega_0$, and let $v \in L^H$. Then the path $p = {}_1\widehat{yv}$ in $\Gamma(X \cup \mathcal{H})$ labelled by the derived word $\widehat{yv}$ is a $(\lambda, c + \lambda + 1)$–quasigeodesic that does not backtrack.*

Proof. Let $v = v_0 s_1 v_1 \cdots s_n v_n$ be the factorisation of v and $\hat{v} = v_0 h_1 \cdots h_n v_n$. We have $y \in \mathrm{Geo}(H, Y)$ by Lemma 4.4(d) and, since $y \notin (X^\pm \cap H_\omega)^*$ for any $\omega \neq \omega_0$ and the first letter x of v does not lie in Y, the word yx is not in any $(X^\pm \cap H_\mu)^*$. Thus the factorisation of yv is $yv_0 s_1 v_1 \cdots s_n v_n$, and $\widehat{yv} = h\hat{v}$, where y represents $h \in H$. Since, by Lemma 4.6, $\hat{v}$ labels a (λ, c)–quasigeodesic path, the path $p := {}_1\widehat{yv}$ is a $(\lambda, c + \lambda + 1)$–quasigeodesic.

Suppose that p backtracks. Then for some ω there are two H_ω–components of p whose initial vertices lie in the same coset of H_ω and, since by Lemma 4.6 the subpath ${}_h\hat{v}$ of p is without backtracking, one of those two components must be the first edge $e := {}_1h$ of p. By our choice of y, the edge e is an H–component of p but not an H_ω–component for any other index $\omega \neq \omega_0$, and so, for some index i, the edge ${}_{h\cdots v_{i-1}}h_i$ of p also has initial vertex in the same coset H as the initial vertex 1 of e, and $s_i =_G h_i$ represents an element of H. But then the nonempty prefix $v_0 s_1 \cdots v_{i-1} s_i$ of the geodesic v represents an element of H and so, by Lemma 4.4(d), this nonempty prefix lies in $(Y^\pm)^*$, contradicting the fact that $v \in L^H$. □

Returning to the proof of Proposition 4.5, we next apply these two lemmas to establish the fellow traveller property. Suppose that $u, v \in L^H$, $x \in X^\pm \cup \{\epsilon\}$, and $h \in H$ satisfy $ux =_G hv$. Let $y \in \text{Geo}(G, X)$ be a geodesic representative of h. Then, by Lemma 4.4, we have $y \in \text{Geo}(H, Y)$. If h is in the finite set $\cup_{\omega\neq\omega_0}(H \cap H_\omega)$ then it follows from the definition of the generating set X of G that $y \in X^\pm \cup \{\epsilon\}$, and so $|y| \leq 1$.

Suppose, on the other hand, that h does not lie in H_ω for any $\omega \neq \omega_0$. Then, by Lemma 4.6 applied to u, the path $p := {}_1\hat{u}$ in $\Gamma(G, X \cup \mathcal{H})$ is a (λ, c)–quasigeodesic without backtracking that does not penetrate the coset H. Since increasing the constants preserves the quasigeodesic property, p is also a $(\lambda, c + \lambda + 1)$–quasigeodesic. By Lemma 4.7, the path $q := {}_1\widehat{yv}$ is a $(\lambda, c + \lambda + 1)$–quasigeodesic as well. Since $h \neq 1$, the path q penetrates the coset H in its first edge ${}_1h$. Moreover, the paths p and q both start at 1, and the group elements at their terminal vertices, represented by u and yv, are connected by a single edge labelled x in $\Gamma(G, X)$. Now, by the Bounded Coset Penetration property of Remark 4.1, the distance between 1 and $h =_G y$ in $\Gamma(G, X)$ is at most the constant $B(\lambda, c + \lambda + 1)$.

So in either case we have $|y| = |h|_X \leq M := \max\{1, B(\lambda, c + \lambda + 1)\}$. Now, by a standard argument, if K is the fellow-traveller constant of the biautomatic structure L for G over X, then the paths ${}_1u$ and ${}_hv$ synchronously K'–fellow travel, with $K' := MK + M$. This completes the proof of (2).

Finally we turn to the crossover property. Suppose that $u, v \in L^H$, $\omega \in \Omega$, $g \in H_\omega$, and $h \in H$ satisfy $ug =_G hv$, where u does not represent an element of H. Let x and y be elements of $\text{Geo}(G, X)$ representing g and h, respectively, and note from Lemma 4.4 that $x \in \text{Geo}(H_\omega, X \cap H_\omega)$ and $y \in \text{Geo}(H, Y)$. If h is in the finite set $\cup_{\omega\neq\omega_0}(H \cap H_\omega)$ then, as above, we have $|y| = |h|_X \leq 1$.

Suppose instead that h is not in this finite set. Then as above, Lemmas 4.6 and 4.7 show that the path $p := {}_1\hat{u}$ is a (λ, c)–quasigeodesic without vertex backtracking that does not penetrate the coset H, and the path $q := {}_1\widehat{yv}$ is a $(\lambda, c + \lambda + 1)$–quasigeodesic without backtracking. Now consider the path $p' := {}_1\hat{u}g$, where we consider g to be a single letter in the generating set $\mathcal{H}$. The path p' is also a $(\lambda, c + \lambda + 1)$–quasigeodesic, since it consists of the path p together with one more edge $e := {}_{\hat{u}}g$. Since p does not penetrate H, and the word u does not represent an element of H, the initial vertex of e is not in H, and so p' also does not penetrate the coset H.

If the path p' does not backtrack then, by the Bounded Coset Penetration property of Remark 4.1, we have $|y| = |h|_X \leq B(\lambda, c + \lambda + 1)$.

Suppose instead that p' does backtrack; then the final edge e of p' penetrates the same $H_{\omega'}$-coset $uH_{\omega'}$ as one of the edges of p, for some index ω', and since

$g \in H_\omega$, we may take $\omega' = \omega$. Let $u = u_0 s_1 u_1 \cdots s_n u_n$ be the factorisation of u, and $\hat{u} = u_0 h_1 \cdots h_n u_n$, and suppose that the edge of p labelled by h_k penetrates the coset uH_ω. Then the suffix $s_k u_k \cdots s_n u_n$ of u represents an element of H_ω and then by Lemma 4.4(d) this suffix is a word in $\mathrm{GEO}(H_\omega, X \cap H_w)$, and so we must have $k = n$ and $u_n = \epsilon$.

So $s_n g$ represents an element $h' \in H_\omega$, and the path p'' labelled by $u_0 h_1 \cdots s_{n-1} u_{n-1} h'$ is a $(\lambda, c + \lambda + 1)$–quasigeodesic without backtracking that does not penetrate H, and with the same initial and terminal vertices as q. Now we can apply Remark 4.1 as before to the paths p'' and q to conclude that $|h|_X \leq B(\lambda, c+\lambda+1)$.

Hence L^H has M-maximal crossover with respect to $(Y, X \cap H_\omega)$, where $M = \max\{1, B(\lambda, c + \lambda + 1)\}$. □

4.3 Synchronous automatic structures for graphs of relatively hyperbolic groups

The following is now an immediate corollary of Proposition 4.5 and Theorems B and 2.2.

Theorem 4.8. *Let $\mathcal{G} = (\Lambda = (V, \vec{E}), \{G_v : v \in V\}, \{G_e : e \in \vec{E}\}, \{\phi_e : e \in \vec{E}\})$ be a graph of groups over a finite connected directed graph Λ. Suppose that the following conditions hold.*

- *(i) Each vertex group G_v is finitely generated and hyperbolic relative to a collection of subgroups, and each edge group G_e with $\tau(e) = v$ is one of those peripheral subgroups.*
- *(ii) Any finite generating set of any peripheral subgroup H of a vertex group G_v can be extended to one over which the peripheral subgroup has a geodesic biautomatic structure.*
- *(iii) For each edge e, the triple $(G_e, G_{\overline{e}}, \phi_e)$ is 1-stable with respect to $(X_{\tau(e)} \cap G_e, X_{\tau(\overline{e})} \cap G_{\overline{e}})$ where for each $v \in V$ the set X_v is a finite generating set for G_v satisfying properties (1)-(3) of Proposition 4.5.*

Then for each edge $e_0 \in \vec{E}$ the pair $(\pi_1(\mathcal{G}), G_{e_0})$ is strongly synchronously coset automatic. Moreover, the fundamental group $\pi_1(\mathcal{G})$ is automatic.

Once again, we observe that the condition (ii) holds in particular when each subgroup H_ω is either virtually abelian (by [3, Prop 10.1]) or hyperbolic.

In general we cannot dispense with the 1-stability assumption in condition (iii) of this theorem even in the case that the vertex groups are hyperbolic relative to abelian subgroups, as the following example shows.

Example 4.9. Let $G = \langle a, b, c \mid ab = ba \rangle \cong \mathbb{Z}^2 * \mathbb{Z}$. Then G is hyperbolic relative to $\{H\}$ with $H := \langle a, b \rangle$. Let $\mathcal{G}$ be the graph of groups with a single vertex, and a single edge e from the vertex group G to itself (so $\pi_1(\mathcal{G})$ is an HNN extension). We define $\phi_e : H \to H$ by $\phi_e(a) = ab$, $\phi_e(b) = b$. Then the resulting fundamental group is isomorphic to $K * \mathbb{Z}$, where K is the Heisenberg group. Since K is not automatic by [16, Theorem 8.1.3], the group $K * \mathbb{Z}$ is not automatic by [16, Theorem 12.1.8].

However, in Corollary 4.10 we show that, in the case when the peripheral subgroups are abelian and have sufficiently limited interaction, we can dispense with the 1-stability assumption in Theorem 4.8.

Corollary 4.10. *Let $\mathcal{G}$ be a graph of groups associated with a finite connected graph Λ and finitely generated vertex groups that are hyperbolic relative to abelian subgroups, and suppose that each edge group is peripheral in its adjacent vertex group. Suppose further that Λ contains no nonempty directed circuit p for which, whenever $e \cdot f$ is a pair of consecutive edges in p, the edge groups corresponding to the terminal vertex of e and the initial vertex of f are equal. Then $\pi_1(\mathcal{G})$ is an automatic group with respect to a Higgins language of normal forms.*

Proof. Let $\mathcal{G} = (\Lambda = (V, \vec{E}), \{G_v : v \in V\}, \{G_e : e \in \vec{E}\}, \{\phi_e : e \in \vec{E}\})$ be this graph of groups. In [16, Theorem 4.3.1], it is shown that every finitely generated abelian group is shortlex automatic over every generating set; moreover, the structure is also biautomatic.

For each $v \in V$, let $X_{v,1}$ be a finite generating set of G_v, let $\{H_{v,\omega} \mid \omega \in \Omega_v\}$ be the collection of peripheral subgroups for G_v, and let $\mathcal{H}_v := \cup_{\omega\in\Omega_v}(H_{v,\omega} \setminus \{1\})$. Let $\mathcal{H}'_v$ be the finite subset of $\mathcal{H}_v$ associated to G_v and $X_{v,1}$ from Lemma 4.4, and let $X_{v,2} := X_{v,1} \cup \mathcal{H}'_v$. Then for each $\omega \in \Omega_v$, the set $X_{v,2} \cap H_{v,\omega}$ generates the group $H_{v,\omega}$.

Let $\widehat{P}(\Lambda)$ be the set of all directed paths in Λ of the form $p = e_1 \cdots e_k$ such that $e_{i+1} \neq \overline{e}_i$ and $G_{e_i} = G_{\overline{e}_{i+1}}$ for all $1 \leq i \leq k-1$; that is, the path p in Λ does not backtrack and the (peripheral) edge subgroups in the vertex group $G_{\tau(e_i)} = G_{\iota(e_{i+1})}$ corresponding to the edges e_i and $\overline{e}_{i+1}$ are the same for all i. The hypotheses show that the set $\widehat{P}(\Lambda)$ is a finite set.

For any $p = e_1 \cdots e_k \in \widehat{P}(\Lambda)$, let $\bar{p} := \overline{e}_k \cdots \overline{e}_1$ be the reverse path, let $G_p := G_{e_k}$, and define $\phi_p : G_{e_k} \to G_{\overline{e_1}}$ by $\phi_p := \phi_{e_1} \circ \cdots \circ \phi_{e_k}$. Note that the hypotheses show that $G_{\overline{e}_1} \neq G_{e_k}$. Also define $Y_p := \phi_{\bar{p}}(X_{\tau(\bar{p}),2} \cap G_{\bar{p}})$.

For each vertex v of Λ, let

$$X_v := X_{v,2} \cup (\cup_{p\in\widehat{P}(\Lambda),\tau(p)=v} Y_p).$$

Now let e be any edge of Λ, and let $v := \tau(e)$ and $\widetilde{v} := \iota(e)$. The set $X_v \cap G_e$ is again a generating set for the peripheral subgroup G_e, over which G_e is geodesic biautomatic. The proof of Proposition 4.5 shows that the generating set X_v of G_v satisfies properties (1)-(3) (with respect to the pair (G_v, G_e)) of that Proposition.

The fact that $H_{v,\omega} \cap H_{v,\mu} \subseteq X_{v,2}$ for all distinct $\omega, \mu \in \Omega_v$ implies that

$$\begin{aligned} X_v \cap G_e &= \Big(X_{v,2} \cup (\cup_{p\in\widehat{P}(\Lambda),\tau(p)=v} Y_p)\Big) \cap G_e \\ &= (X_{v,2} \cap G_e) \cup Y_e \cup (\cup_{p\in\widehat{P}(\Lambda)\setminus\{e\},G_p=G_e} Y_p), \end{aligned}$$

and similarly

$$X_{\widetilde{v}} \cap G_{\overline{e}} = (X_{\widetilde{v},2} \cap G_{\overline{e}}) \cup Y_{\overline{e}} \cup (\cup_{q\in\widehat{P}(\Lambda)\setminus\{\overline{e}\},G_q=G_e} Y_q).$$

Now $\phi_e(X_{v,2} \cap G_e) = Y_{\overline{e}}$ and $\phi_e(Y_e) = \phi_e(\phi_{\overline{e}}(X_{\widetilde{v},2} \cap G_{\overline{e}})) = X_{\widetilde{v},2} \cap G_{\overline{e}}$. Suppose that $p \in \widehat{P}(\Lambda) \setminus \{e\}$ satisfies $G_p = G_e$. If the last edge of the path p is e, then we can write $p = q \cdot e$ for some path $q \in \widehat{P}(\Lambda) \setminus \{\overline{e}\}$ satisfying $G_q = G_{\overline{e}}$, and so

$$\phi_e(Y_p) = \phi_e(\phi_{\bar{p}}(X_{\tau(\bar{p}),2} \cap G_{\bar{p}})) = \phi_e((\phi_{\overline{e}} \circ \phi_{\bar{q}})(X_{\tau(\bar{p}),2} \cap G_{\bar{p}})) = Y_q.$$

On the other hand, if the last edge of p is not e, then the path $q := p \cdot \overline{e}$ lies in $P(\Lambda) \setminus \{\overline{e}\}$ and satisfies $G_q = G_{\overline{e}}$, and the argument in the previous sentence shows that $\phi_{\overline{e}}(Y_q) = Y_p$; hence $\phi_e(Y_p) = Y_q$. Hence ϕ_e maps $X_v \cap G_e$ to $X_{\widetilde{v}} \cap G_{\overline{e}}$. Similarly $\phi_{\overline{e}}$ maps $X_{\widetilde{v}} \cap G_{\overline{e}}$ to $X_v \cap G_e$; that is, ϕ_e is a bijection from the generating set $X_v \cap G_e$ of G_e to the generating set $X_{\widetilde{v}} \cap G_{\overline{e}}$ of $G_{\overline{e}}$. Hence the triple $(G_e, G_{\overline{e}}, \phi_e)$ is 1-stable with respect to this pair of generating sets. The result now follows from Theorems 4.8 and 2.2. □

We already noted in Section 1 that the automaticity of $\pi_1(\mathcal{G})$ in the above result was previously known, with respect to a different normal form. in particular, it follows from Dahmani's Combination Theorem [12, Theorem 0.1] that $\pi_1(\mathcal{G})$ is hyperbolic relative to a family of abelian groups, and then application of [3, Corollary 1.8] shows that $\pi_1(\mathcal{G})$ is shortlex biautomatic.

We can apply Corollary 4.10 to the construction of automatic structures for fundamental groups of 3-manifolds. Although fundamental groups of closed 3-manifolds with JSJ decomposition pieces that do not have Nil or Sol geometry have been shown by Epstein et al. [16, Thm. 12.4.7] to be automatic, the normal forms for the automatic structure are difficult to determine from the construction in that proof. The proofs of Theorems A and 4.8 were partly inspired by the proof in [8] that all fundamental groups of closed 3-manifolds have the related property of autostackability, and as in that earlier proof, our proofs of those theorems use the set of Higgins normal forms described in Section 3.1. We now show that when the pieces of the JSJ decomposition are hyperbolic, the fundamental group of the 3-manifold is also automatic over those normal forms.

Corollary C. *Let M be an orientable, connected, compact 3-manifold with incompressible toral boundary whose prime factors have JSJ decompositions containing only hyperbolic pieces. Then the group $\pi_1(M)$ is automatic, with respect to a Higgins language of normal forms.*

Proof. The manifold M is a connected sum of finitely many prime manifolds, $M = M_1 \# \cdots \# M_k$, and the fundamental group $\pi_1(M)$ is the free product of the groups $\pi_1(M_i)$.

For each index i, the group $\pi_1(M_i)$ is a fundamental group of a graph of finitely generated groups that are hyperbolic relative to (free) abelian subgroups, over a finite connected graph Λ_i. Moreover, this graph of groups satisfies the properties that each edge group is a peripheral subgroup in its vertex group, and for any two edges e, f of Λ_i with the same terminal vertex $\tau(e) = \tau(f)$, the intersection of the corresponding edge groups is $G_e \cap G_f = \{1\}$. Hence conditions (i) and (ii) of Corollary 4.10 are satisfied, and so $\pi_1(M_i)$ is automatic with respect to a Higgins language L_i of normal forms.

The free product $\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_k)$ is automatic with respect to the standard normal form set L of a free product [16, Theorem 12.1.4], constructed using the languages L_i of normal forms for the factor groups above. Then $\pi_1(M)$ is also the fundamental group of a graph of groups built from the graphs of groups defining the groups $\pi_1(M_i)$, by joining the graph Λ_i to the graph Λ_{i+1} by an edge whose associated edge groups are the trivial group for each i, and the language L is a Higgins language for this graph of groups. □

Remark 4.11. For a nonorientable, connected, compact 3-manifold M with incompressible toral boundary, whose JSJ pieces have interiors with hyperbolic geometry, there is an orientable 2-sheeted cover M' of M satisfying the hypotheses of Corollary C, and $\pi_1(M')$ is an index 2 subgroup of $\pi_1(M)$. Hence in this case, by [16, Theorem 4.1.4] and Corollary C, the group $\pi_1(M)$ has an automatic structure with a language that is the concatenation of the Higgins normal forms for $\pi_1(M')$ with a transversal for $\pi_1(M')$ in $\pi_1(M)$.

5 Synchronous automaticity when geodesics concatenate up

In this section we introduce the property for a pair of groups (G, H) that geodesics in G 'concatenate up' from the subgroup H; such a pair (G, H) is known in the literature as an *admissible pair*. In Section 5.1 we study crossover properties for shortlex automatic groups in which geodesics concatenate up from subgroups, and use this to prove that strong synchronous coset automaticity is preserved by the graph of groups construction when geodesics for all edge groups G_e concatenate up to geodesics for their adjacent vertex groups $G_{\tau(e)}$.

Let $G = \langle X \rangle$ be a group and, for some $Y \subseteq X$, let $H = \langle Y \rangle$ be the subgroup of G generated by Y.

Definition 5.1. We say that geodesics for H over Y *concatenate up* to geodesics for G over X (or $\textsc{Geo}(H, Y)$ concatenates up to $\textsc{Geo}(G, X)$) provided that whenever w is a geodesic word over Y and v_0 is a word over X that is a minimal length representative of its coset, the word wv_0 is also geodesic.

Note that this property implies that any element of G has a geodesic representative of this form wv_0.

The property of geodesics concatenating up has been used by Alonso [1] and Chiswell [9] to study the growth functions of amalgamated free products, HNN extensions, and fundamental groups of graphs of groups. Examples of subgroups in groups with generating sets for which geodesics concatenate up include any sub-graph product of a graph product of groups (including a direct factor in a direct product, or a free factor in a free product) [10], [23, Prop. 14.4]. Alonso's article [1] provides many other examples.

In Section 5.2 we prove that Coxeter groups and sufficiently large Artin groups have the property of geodesics concatenating up with respect to special subgroups (over the standard Coxeter and Artin generating sets), and hence amalgamated products, HNN extensions, and more generally fundamental groups of graphs of these groups over parabolic subgroups are automatic.

5.1 Crossover and strong sychnonous coset automaticity for graphs of groups when geodesics concatenate up

In order to obtain, in Theorem D, SSCA for fundamental groups of graphs of groups in the case that geodesics concatenate up, we begin by describing a situation that yields a geodesic SSCA and a 1-limited crossover condition for a subgroup in a group.

Proposition 5.2. *Let $G = \langle X \mid R \rangle$, let $H = \langle Y \rangle$ for some $Y \subset X$, and suppose that geodesics for H over Y concatenate up to geodesics for G over X. Suppose that G is shortlex automatic with respect to some ordering of $X^{\pm}$ in which all* *elements of $Y^{\pm}$ precede all elements of $X^{\pm} \setminus Y^{\pm}$, and let the languages* SL *and* SL^H *be defined with respect to that ordering. Then the coset language* SL^H *has* 1*-limited crossover with respect to (Y, Z) for any $Z \subseteq X$, and defines a strong synchronous (shortlex) automatic coset system for (G, H).*

Proof. Let $u \in \mathrm{SL}^H$ and $x \in X^{\pm}$. The conclusions will follow once we have proved that either $ux =_G v$ with $v \in \mathrm{SL}^H$ or $ux =_G yv$ with $v \in \mathrm{SL}^H$ and $y \in Y^{\pm}$. In the first case, since u and v are both in SL^H, we must have $||u|-|v|| \leq 1$. In the second case, we shall show that $|u| = |v|$. These restrictions on u and v will be used later in the proof of Proposition 6.4.

Note that v or yv will then be proved to be the unique representative of ux in SL, since we put all the letters of $Y^{\pm}$ first in the ordering, provided that in the case where there is more than one choice for yv we choose that one with y earliest in the ordering of $Y^{\pm}$. So the synchronous fellow travelling of the path ${}_1u$ with the path ${}_1v$ or ${}_yv$ then follows from the (synchronous) shortlex automaticity of G.

Case 1. Suppose first that the word ux is not geodesic, and let $v_1 \in \mathrm{SL}$ represent the element ux of G. Then $|v_1|$ is equal to either $|u| - 1$ or $|u|$. We claim that $v_1 \in \mathrm{SL}^H$, which will complete the proof in this case.

If not, then let v_0 be the representative of Hv_1 in SL^H, so $v_0 <_{\mathrm{SL}} v_1$, with $v_1 =_G hv_0$ for some $h \in H$. Since geodesics for H concatenate up to G, whenever w is a geodesic representative for h, the word wv_0 must be geodesic. It follows that $|v_0| < |v_1|$. But now $u =_G v_1x^{-1} =_G hv_0x^{-1}$ with $|v_0x^{-1}| < |v_1| + 1 \leq |u| + 1$, and so $v_0x^{-1} =_G h^{-1}u$. Now (since geodesics concatenate up) $w^{-1}u$ must be geodesic, but we have $|v_0x^{-1}| < |u| + 1 \leq |w^{-1}u|$, and so we have a contradiction.

Case 2. Suppose now that ux is geodesic. Let $ux =_G v_1$ with $v_1 \in \mathrm{SL}$ representing the element ux of G. So $|v_1| = |ux|$. If $v_1 \in \mathrm{SL}^H$ then we are done.

If not, then again let v_0 be the representative of v_1 in SL^H, so $v_0 <_{\mathrm{SL}} v_1$ with $v_1 =_G hv_0$ for some $h \in H$. Let w be a geodesic representative of h. Then, again, since geodesics concatenate up, wv_0 must be geodesic, of the same length as v_1, and so $|v_0| < |v_1|$ and $|w| = |v_1| - |v_0|$.

If $|v_0| = |v_1| - 1$, then $|w| = 1$, so $w \in Y^{\pm}$, and so $ux =_G wv_0$ with $|v_0| = |v_1| - 1 = |u|$, which proves the result.

Otherwise, $|v_0| \leq |v_1| - 2$. Then $|w| \geq 2$, and $ux =_G v_1 =_G wv_0$. Then $v_0x^{-1} =_G w^{-1}u$, and, since geodesics concatenate up, $w^{-1}u$ must be geodesic. But $|w^{-1}u| > |v_0x^{-1}|$, so we have a contradiction. This contradiction completes the proof of the proposition. □

Theorem D. *Let $\mathcal{G} = (\Lambda = (V, \vec{E}), \{G_v : v \in V\}, \{G_e : e \in \vec{E}\}, \{\phi_e : e \in \vec{E}\})$ be a graph of groups over a finite connected graph Λ. Let X_v and Y_e be finite generating sets of the groups G_v and G_e, respectively. Suppose that the following conditions hold for each $e \in \vec{E}$.*

- *(i)* $Y_e \subseteq X_{\tau(e)}$.
- *(ii)* $\mathrm{Geo}(G_e, Y_e)$ *concatenates up to* $\mathrm{Geo}(G_{\tau(e)}, X_{\tau(e)})$.
- *(iii) The triple $(G_e, G_{\overline{e}}, \phi_e)$ is 1-stable with respect to $(Y_e, Y_{\overline{e}})$.*
- *(iv) $G_{\tau(e)}$ is shortlex automatic with respect to an ordering of $X_{\tau(e)}$ in which all letters of $Y_e^{\pm}$ precede all letters of $X_{\tau(e)}^{\pm} \setminus Y_e^{\pm}$.*

Let $\mathcal{L}$ be the set of coset languages $\mathrm{SL}^{G_e}_{G_{\tau(e)}}$, for $e \in \vec{E}$, and let $\mathcal{T}$ be any maximal tree in Λ. Then, for each $e_0 \in \vec{E}$, the pair $(\pi_1(\mathcal{G}), G_{e_0})$ is strongly synchronously coset automatic, with the Higgins coset language $L := L(\mathcal{G}, \mathcal{L}, e_0, \mathcal{T})$. Furthermore $L \subseteq \mathrm{GEO}^{G_{e_0}}$, and the group $\pi_1(\mathcal{G})$ is automatic.

Proof. Define $v_0 := \tau(e_0)$, $H := G_{e_0}$ and $X := \cup_{v \in V} X_v \cup \{s_e : e \in \vec{E} \setminus \vec{E}_{\mathcal{T}}\}$. We apply Proposition 5.2 in order to verify for each $e \in \vec{E}$ that the pairs $(G_{\tau(e)}, G_e)$ satisfy conditions (i) and (iii) of Theorem A. Since we are already assuming hypothesis (ii) of that theorem, we can apply it to conclude that the pair (G, H) is SACA, with the language as described.

Our next step is to prove that $L \subseteq \mathrm{GEO}^H$. So suppose that $w \in L$, and let u be a representative of Hw of minimal X-length. We create a word $\hat{u}$ from u by inserting into u symbols s_e for $e \in \vec{E}_{\mathcal{T}}$, so that $\hat{u} = u_0 s_{e_1} u_1 \cdots s_{e_k} u_k$, where $e_1 \cdots e_k$ is a path within Λ that starts at v_0, and where $u_0 \in (X^{\pm}_{\tau(e_0)})^*$ and $u_i \in (X^{\pm}_{\tau(e_i)})^*$ for each i. Note that $\mathrm{Defl}(\hat{u}, \mathcal{T}) = u$. Next we construct an $(\mathcal{L}, \mathrm{SL}_{G_{e_0}} \mathrm{SL}^{G_{e_0}}_{G_{\tau(e_0)}})$–cascade of u, as follows. We define u'_k to be the shortlex minimal representative word over $X_{\tau(e_k)}$ in the coset $G_{e_k} u_k$, and suppose that $u_k =_{G_{\tau(e_k)}} h_k u'_k$. Our hypothesis (ii) ensures that $|u_k| = |u'_k| + |h_k|_{Y_{e_k}}$. Now we define $h'_k \in G_{\overline{e}_k}$ to be the element $\phi_{e_k}(h_k) =_G s_{e_k} h_k s_{e_k}^{-1}$. The 1-stability condition implies that $|h'_k|_{Y_{\overline{e}_k}} \le |h_k|_{Y_{e_k}}$. We repeat this procedure, but using $u_{k-1} h'_k$ rather than u_k, and so define elements $h_k, \ldots, h_1, h_0$, words $u'_k, \ldots, u'_0 \in \mathrm{SL}^{G_{e_j}}_{G_{\tau(e_j)}}$, and elements $h'_k, \ldots, h'_1$. Let w_0 be a geodesic word over Y_{e_0} that represents h_0. The deflation u' of the word $w_0 u'_0 s_{e_1} u'_1 s_{e_2} u'_2 \cdots s_{e_k} u'_k$, which represents the same element as u, is no longer than u. Hence u' must be geodesic, and since the deletion of w_0 results in a word in the same coset, w_0 must be the empty word (and so $h_0 =_G 1$). Now $u' \in L$. Since L has uniqueness, we have $u' = w$.

Now synchronicity of L follows by Proposition 3.9. Then application of Theorem 2.2 proves that $\pi_1(G)$ is automatic. □

5.2 Application to graphs of Coxeter and sufficiently large type Artin groups

We assume that the reader is familiar with the definitions of Coxeter groups and Artin groups (also known as Artin–Tits groups, of which Coxeter groups are natural quotients) and with the presentations of these groups over their standard generating sets; for Coxeter groups, [5] is a standard reference.

The following lemma is noted in [1, Example 1], and an outline of the proof is given in [5, Exercise Ch.IV §1(26)]. It is also proved in [2, Proposition 7.11].

Lemma 5.3. *Let $G = \langle X \rangle$ be a Coxeter group, defined over its standard generating set X, and let $H = \langle Y \rangle$ be a subgroup of G, for some $Y \subseteq X$. Then geodesics for H concatenate up to G.*

Proof. The proof of the lemma uses the *Exchange Lemma* [5, Chapter IV.1.4, Lemma 3] for Coxeter groups, which says that, in any non-geodesic word over X, we can get a shorter representative of the same group element by removing two of the letters in the word.

We let w be a geodesic word over Y, and v a geodesic word over X such that wv is non-geodesic, and prove that in that case v cannot be of minimal length within its coset.

Let $w'v$ be a minimal non-geodesic word with w' a suffix of w. Since v is geodesic, w' is nonempty. Let $w' = tw''$ with $t \in Y^{\pm}$. Then, by the Exchange Lemma, we can remove two of the letters of the non-geodesic word $tw''v$ to get a shorter representative of the same group element. Since $w''v$ and tw'' are both geodesic, one of these removed letters must be t and the other must lie in v. So the result of removing this letter from v is a shorter representative of the coset Hv. $\square$

Given a Coxeter graph Σ (that is, a finite simple graph whose edges are labelled by parameters m_{ij} each from the set $(\mathbb{N} \setminus \{0, 1\}) \cup \{\infty\}$), we denote by A_Σ the associated Artin group. Suppose that G is an Artin group with standard generating set X, and that the integers m_{ij} are the parameters of the standard presentation (which label the edges of Σ). The group G, as well as the Coxeter graph Σ, is said to be of *large type* if for all $i \neq j$, $m_{ij} \geq 3$, and (following [22]) of *sufficiently large type* if for any triple i, j, k either none of m_{ij}, m_{ik}, m_{jk} are equal to 2, or all three of m_{ij}, m_{ik}, m_{jk} are equal to 2, or at least one m_{ij}, m_{ik}, m_{jk} is infinite.

The following lemma is proved in [10] (see also [23, Prop. 14.4]) for the special case of right-angled Artin groups. In order to prove the result for Artin groups of sufficiently large type, we need to use knowledge of geodesics in these groups, and of a process that reduces any word to geodesic form, which is described in [21, 22]. In particular, some familiarity with the concept of *critical sequences of moves* applied to words over the generators is required in order to understand the following proof.

Lemma 5.4. *Let $G = \langle X \rangle$ be an Artin group of sufficiently large type, defined over its standard generating set X, and let $H = \langle Y \rangle$ be a subgroup of G, for some $Y \subseteq X$. Then geodesics for H concatenate up to G.*

Proof. We prove the contrapositive, as follows. As in the proof of Lemma 5.3, we let w be a geodesic word over Y, and v a geodesic word over X such that wv is non-geodesic, and prove that in that case v cannot be of minimal length within its coset. We prove this by showing that v must be equal in G to a geodesic word that starts with a letter of $Y^{\pm}$.

Let $w'v$ be a minimal non-geodesic word with w' a suffix of w. So, if u is a geodesc word u with $u =_G w'v$, then $|w'v| - |u| \leq 2$ but, since all defining relators of Artin groups have even length, we must have $|u| = |w'v| - 2$. Since v is geodesic, w' is nonempty. Let $w' = tw''$ with $t \in Y^{\pm}$. Then $w''v =_G t^{-1}u$ with both words geodesic.

If w'' is empty then $v =_G t^{-1}u$, which proves the lemma. Otherwise, since w is geodesic, w'' does not start with t^{-1}, but it starts with a letter in $Y^{\pm}$. By [22, Prop 3.2 (1)] (applied with 'left' in place of 'right'), a single leftward critical sequence (a sequence of overlapping replacements of 2-generator subwords by words of the same length on the same 2 generators) can be applied to $w''v$ to transform it to a word starting with t^{-1}. The moves in the sequence cannot all take place within w'' because that would contradict w being geodesic. If some of the moves in the critical sequence take place within v, then we can just change

v to the result of these moves. So we can assume that the first move $\mu_1 \to \nu_1$ in the sequence overlaps both w'' and v.

We claim that the two generators involved in this move must both be in Y. So suppose that one of them, s say, does not. The first letter of μ_1 lies in w'' and hence in $Y^{\pm}$, and so ν_1 begins with s or s^{-1}. If there is a second move $\mu_2 \to \nu_2$ in the sequence, then μ_2 begins with a letter in w'' and hence in $Y^{\pm}$ and ends with $s^{\pm 1}$, so ν_2 must also begin with s or s^{-1}. Then we see by induction that, for all moves $\mu_i \to \nu_i$ in the sequence, ν_i begins with s or s^{-1} and hence, after applying the complete sequence, the resulting word begins with s or s^{-1}. But we know already that it begins with t^{-1} with $t \in Y^{\pm}$, so we have a contradiction, which proves the claim.

Since one of the two generators involved in the first move $\mu_1 \to \nu_1$ is the first letter of v or its inverse, it follows that the first letter of v is in $Y^{\pm}$, and the lemma is proved. $\square$

Corollary 5.5. *A fundamental group of a graph of groups in which each vertex group is either a Coxeter group or a sufficiently large type Artin group, and in which each edge group is a special subgroup in its adjacent vertex group, is automatic.*

We note that a fundamental group of such a graph of groups built only out of Coxeter groups, or only out of Artin groups, must itself be such a group. And conversely, by [24], any Coxeter group that arises as the fundamental group of a graph of groups must arise in a similar way.

The following gives a number of examples.

Corollary E. *Let Σ be a Coxeter graph of sufficiently large type. Given arbitrary subgraphs $\Lambda_1, \Lambda_2, \ldots, \Lambda_k$ of Σ, suppose that the Coxeter graph Σ' is formed by adjoining new vertices $v_1, v_2, \ldots, v_k$ to Σ together with the following edges from each v_i:*

to each vertex of Λ_i, with the label 2,
to each vertex of $\Sigma \setminus \Lambda_i$, with the label ∞,
to each vertex v_j with $j \neq i$, with the label ∞.

Then the Artin group $A_{\Sigma'}$ is automatic.

Proof. The Artin group $A_{\Sigma'}$ is a multiple HNN-extension of $G_v = A_\Sigma$ over the subgroups $G_{e_i} = A_{\Lambda_i}$, where $\phi_{e_i} \colon A_{\Lambda_i} \to A_{\Lambda_i}$ is the identity map. Thus, this graph of groups satisfies condition (iii) of Theorem D. Further, $X_v = V(\Sigma)^{\pm}$ and $Y_{e_i} = V(\Lambda_i)^{\pm}$ and so condition (i) is satisfied. By Lemma 5.4, geodesics from A_{Λ_i} concatenate up to A_Σ, and so condition (ii) is satisfied. Condition (iv) follows from [22]. Thus, $(A_{\Sigma'}, A_{\Lambda_1})$ is SSCA. Moreover, A_{Λ_i} is also shortlex automatic, by [22]. Thus, by Theorem 2.2, $A_{\Sigma'}$ is automatic. $\square$

Example 5.6. A 4-generator example is provided by extending the Artin group of type $\widetilde{A_2}$ by one generator y_1, defined to commute with two of the existing generators. This is the Artin group defined by the Coxeter diagram shown in Figure 5.

As far as the authors know, automaticity of the family of Artin groups covered by Corollary E was previously unknown, since this family includes groups that are not of sufficiently large type. On the other hand, it was clear that

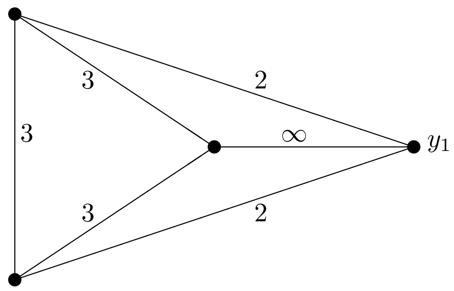

Figure 5: An Artin group not previously known to be automatic

(as fundamental groups of graphs of groups) they had solvable word problem, though quadratic Dehn function was (probably) unknown.

Further, it can be shown that these Artin groups satisfy Dehornoy's property H, introduced in [13], which implies that their word problem is solvable via padded multifraction reduction [14, Proposition 1.14].

6 Further strong synchronous coset automatic structures

For our next example of a family of groups and subgroups with limited crossover, we consider the case in which the group is abelian. It is proved in [16, Theorem 4.3.1] that finitely generated abelian groups are shortlex automatic over all finite generating sets. The following proposition expands the result to coset systems relative to any subgroup.

Proposition 6.1. *Let $G = \langle X \rangle$ be a finitely generated abelian group, and let $H = \langle Y \rangle$ be a subgroup. Then (G, H) is strongly synchronously coset automatic with 1-limited crossover with respect to Y. Furthermore, for any ordering of $X^{\pm}$, we can choose the coset automatic structure to consist of the shortlex least representatives of the cosets of H.*

Proof. We suppose that $X^{\pm} = \{x_1, \ldots, x_n\}$, with $x_1 < x_2 < \ldots < x_n$. For each $x \in X \cup X^{-1}$, write $\bar{x}$ for the coset Hx. Then $\bar{X} := \{\bar{x} : x \in X\}$ is a generating set for $\bar{G} := G/H$, and each word w over X has an image $\bar{w}$ over $\bar{X}$. Let $\rho : X \to \bar{X}$ be the map $x \mapsto \bar{x}$. Note that ρ might not be injective, but we may choose an injective map $\rho' : \bar{X} \to X$ such that $\rho\rho'(\bar{x}) = \bar{x}$ for all $\bar{x} \in \bar{X}$, and then extend ρ' to an injective monoid homomorphism from words over $\bar{X}^{\pm}$ to words over $X^{\pm}$.

By [16, Theorem 4.3.1], $\bar{G}$ is shortlex automatic with respect to the generating set $\bar{X}$; we define $L_{\bar{G}}$ to be the shortlex language for $\bar{G}$. Now we choose L to be the set $\rho'(L_{\bar{G}})$. Then, as the image of a regular set under a monoid homomorphism, L is regular. The words in $L_{\bar{G}}$ all have the form $\bar{x}_1^{r_1} \cdots \bar{x}_n^{r_n}$, with $r_i \geq 0$, and those in L the form $x_1^{r_1} \cdots x_n^{r_n}$.

Now suppose that $v = x_1^{a_1} \cdots x_n^{a_n}$, $w = x_1^{b_1} \cdots x_n^{b_n}$ are elements of L and $h \in H$ with $d_{\Gamma(G)}(v, hw) \leq 1$. Then $d_{\Gamma(\bar{G})}(\bar{v}, \bar{w}) \leq 1$, and it follows from the proof of [16, Theorem 4.3.1] that $\Sigma_i |b_i - a_i| \leq B$, for some constant B. Hence, since $h =_G vxw^{-1}$, with $x \in X \cup X^{-1} \cup \{\epsilon\}$, the length of h is bounded by

$B+1$. We can also see that, where $v(j), w(j)$ denotes the prefixes of v, w of length j, each of the elements $v(j)^{-1}w(j)$ is represented by a product $x_1^{r_1}\cdots x_n^{r_n}$ with $\Sigma|r_i| \le 2B$. It follows that the differences $v(j)^{-1}hw(j) =_G hv(j)^{-1}w(j)$ are bounded in length.

Since G is abelian, it is straightforward to show that L has limited crossover with respect to the pair (Y, Y). $\square$

Proposition 6.2. *Let G be a finitely generated virtually abelian group, and let H be a subgroup. Then (G, H) is strongly synchronously coset automatic.*

Proof. We shall construct first a coset language L_1^H for G. We have already considered the case when G itself is abelian in Proposition 6.1. It will be convenient here, however, in this special case to define a second language L_2^H, which has additional properties that we shall need in the proof of Proposition 6.5.

Let $F \trianglelefteq G$ with F free abelian and $|G : F|$ finite. By Lemma 2.8(iii) it is sufficient to prove our result for the subgroup FH of G. But it will not be convenient to make that assumption in the aforementioned application to Proposition 6.5 so, in the case when G is nonabelian, we shall assume from now on that $G = FH$, but not when G is abelian.

Let $J := H \cap F$. Then, since either $G = FH$ or G is abelian, we have $J \trianglelefteq G$. We can find a subgroup $E \ge J$ of F such that E/J is torsion-free, and E is characteristic of finite index in F, and hence normal of finite index in G. (We can define $E \le F$ as the inverse image under the natural map $F \to F/J$ of the e-th power of a complement in F/J of the torsion subgroup T of F/J, where e is the exponent of T.)

If G is abelian, then we choose C to be any complement of J in E. Otherwise we apply Lemma 6.3 below with $\widehat{G} = G/E$ to the $\mathbb{Z}G/E$-module E and its submodule J; the submodule U guaranteed by the lemma corresponds to a normal subgroup C of G within E, with $J \cap C = 1$ and $|E : JC|$ finite. Then, in either case, the free abelian group JC is a direct product $J \times C$, and has finite index in G with J and C both normal in G.

We shall define both of our coset languages with respect to a finite generating set X for G that is a union $X_J \cup X_C \cup X_T$, where X_J and X_C are finite generating sets for J and C, and X_T is a set of (not necessarily unique) representatives of the nontrivial cosets of JC in G, satisfying the condition that whenever a coset has nontrivial intersection with H, the representatives in X_T are all within H.

We describe first the construction of L_1^H. The quotient G/J is virtually abelian with free abelian subgroup JC/J of finite index. For each $g \in G$, write $\bar{g}$ for the coset Jg. By [16, Proof of Corollary 4.2.4], there is an automatic structure (actually a geodesic biautomatic structure with uniqueness) for $\bar{G} := G/J$ with language $L_{\bar{G}}$ consisting of words over a finite generating set Z for $\bar{G}$ of the form $Z_C \cup Z_T$, where Z_C is a particular generating set for $\overline{CJ}$ and Z_T is a set of (unique) representatives of the nontrivial cosets of $\overline{CJ}$ in $\bar{G}$, satisfying the condition that whenever a coset has nontrivial intersection with $\bar{H}$ its representative is chosen to be in $\bar{H}$. We let K be the fellow traveller constant associated with this automatic structure. The subsets X_C, X_T of G that we need to define X are chosen to be subsets of G that map bijectively under the map $g \mapsto \bar{g}$ to Z_C, Z_T, and such that $X_C \subseteq C$, while X_J can be any finite generating set of J. So we have a bijection $\rho : X_C \cup X_T$ to $Z_C \cup Z_T$, and we extend ρ^{-1} to a monoid homomorphism that maps words over $Z_C \cup Z_T$ to the corresponding words over $X_C \cup X_T$.

The language $L_{\bar{G}}$ is defined in [16, Proof of Corollary 4.2.4] to be a set of words of the form $\bar{w}$ or $\bar{w}\bar{t}$, where $\bar{w}$ is a word over Z_C, and $\bar{t} \in Z_T$. We define $L_1^H := \rho^{-1}(L_{\bar{G}})$. So, as the image of a monoid homomorphism, L_1^H is regular, and its elements have the form wt, where w is a word over X_C and $t \in X_T \cup \{\epsilon\}$. We observe also that the set Z_C and also the set of words $\bar{w}$ that arise in this language are invariant under conjugation by elements of $\bar{G}$.

We claim that the language L_1^H is a strong automatic coset system for (G, H). We have seen that it is regular, and it contains a full set of coset representatives of H in G (recall that $J \subseteq H$).

It remains to prove the fellow traveller property. So suppose that $w_1t_1, w_2t_2 \in L_1^H$, and $x \in X \cup X^{-1} \cup \{1\}$, $h \in H$ with

$$w_1t_1x =_G hw_2t_2. \quad (*)$$

We need to show that the paths ${}_1w_1$ and ${}_hw_2$ fellow travel (and hence so do ${}_1w_1t_1x$ and ${}_hw_2t_2$).

Our first step towards this proof is to define $c_1 \in C$, $j_1 \in J$ and $t_3 \in X_T$ such that $t_1x =_G j_1c_1t_3$. Since there are only finitely many possible choices for each of t_1, and x, we see that c_1 and j_1 are bounded in length. Let B be an upper bound on their lengths. Now we find $w_3 \in L_1^H \cap (X_C^{\pm})^*$, with $w_1c_1 =_G w_3$ (and so $w_1j_1c_1 =_G j_1w_3$) and, since $L_{\bar{G}}$ is an automatic structure for $\bar{G}$, we see that ${}_1w_1$ and ${}_1w_3$ fellow travel at distance $K|c_1| \le KB$. We now have $w_1t_1x =_G j_1w_3t_3$.

Now we consider the right hand side hw_2t_2 of the equation $(*)$. We can find $j' \in J, t_4 \in X_T$, with $h =_G j't_4$ and so $hw_2t_2 =_G j't_4w_2t_2 =_G j't_4w_2t_4^{-1}t_4t_2$. As we observed above, the generating set Z_C of $\overline{CJ}$ is closed under conjugation by elements of $\bar{G}$ and so, for each generator y that occurs in the word w_2, the image $t_4yt_4^{-1}$ in G/J is in Z_C. The normality of C in G ensures that $t_4yt_4^{-1}$ is a generator in the set X_C that consists of inverse images under ρ of the elements of Z_C. Let w_4 be the word formed from w_2 by replacing each of its generators y by the generator in X_C that represents $t_4yt_4^{-1}$. Then $w_4 =_G t_4w_2t_4^{-1}$ and, by the invariance property of the language $L_{\bar{G}}$ mentioned earlier, the image of w_4 in $\bar{G}$ lies in $L_{\bar{G}}$. Now we define $c_2 \in C, j_2 \in J$ (also each bounded above in length by B), and $t_5 \in X_T$ such that $t_4t_2 =_G j_2c_2t_5$, and hence $j't_4w_2t_2 =_G j'w_4j_2c_2t_5 =_G j'j_2w_5t_5$, where $w_5 =_G w_4c_2$. Just as for the words ${}_1w_1$ and ${}_1w_3$ discussed above, we find that ${}_1w_4$ and ${}_1w_5$ fellow travel at distance KB.

Now recall that we have $w_1t_1x =_G hw_2t_2 \quad (*)$. The left hand side of $(*)$ is equal in G to $j_1w_3t_3$, and the right hand side to $j'j_2w_5t_5$, Since X_T is a transversal of JC in G we have $t_3 = t_5$ and, since JC is a direct product of C and J, we have $w_3 =_G w_5$ and $j_1 =_G j'j_2$. Since $L_{\bar{G}}$ is an automatic structure with uniqueness, we also have $w_3 = w_5$ (as words). Further, j' is bounded in length by $2B$.

So, since the pairs of words $({}_1w_1, {}_1w_3)$, and $({}_1w_4, {}_1w_5) = ({}_1w_4, {}_1w_3)$ both KB-follow travel, to complete our proof it suffices to show that ${}_1w_4$ and ${}_hw_2$ fellow travel. We recall that for generators $a_1, \ldots, a_n, b_1, \ldots, b_n \in X_C^{\pm 1}$ with $b_i =_G t_4a_it_4^{-1}$, we have $w_2 = a_1 \cdots a_n$ and $w_4 = b_1 \cdots b_n$. For each i, the word difference $(a_1 \cdots a_i)^{-1}t_4(b_1 \cdots b_i)$ is equal in G to t_4^{-1}, and so ${}_1w_4$ and ${}_{t_4}w_2$ 1-fellow travel. Since $h =_G j't_4$, j' is bounded in length and JC is abelian, it follows that ${}_1w_4$ and ${}_hw_2$ fellow travel, and we are done.

We turn now to the definition of our second synchronous automatic coset system L_2^H in the case when G is abelian. In this case, we allow X_T to be any finite set of elements from $G \setminus JC$ that contains at least one representative of each nontrivial coset of JC in G. Further, the conditions on the generating sets X_J and X_C of J and C are different from those of L_1^H; they are chosen to ensure that for all equations of the form $t_1t_2t_3 =_G jct_4$ with $t_1, t_2, t_3, t_4 \in X_T^{\pm}$, $j \in J$ and $c \in C$, the elements j and c and included in X_J and X_C. (This property is not used in the current proof, but it is required in the proof of Proposition 6.5 below.)

For our coset language L_2^H, we take the set of words of the form wv, where $w \in \mathrm{SL}(C, X_C)$, and v is a word of length at most 2 over X_T. This language is regular, as it is the concatenation of two regular languages, and this language contains representatives of all cosets of H within G.

It remains to prove the fellow traveller property, and we can do this very much as we did for L_1^H. We suppose that $w_1v_1, w_2v_2 \in L_2^H$ with $w_1, w_2 \in \mathrm{SL}(C, X_C)$ and v_1, v_2 words of length at most 2 over X_T, and $x \in X \cup X^{-1} \cup \{1\}$, $h \in H$ with

$$w_1v_1x =_G hw_2v_2. \quad (*)$$

We can write $h = j't$ with $j' \in J$ and $t \in X_T \cup \{1\}$, and so $w_1v_1x =_G j'w_2tv_2$. Then $JCv_1x = JCtv_2$, so $v_1x =_G jctv_2$, for some $j \in J$ and $c \in C$, and hence $j' =_J j$ and $w_1c =_C w_2$. Since there are only finitely many possible v_1, v_2 and t, the lengths of j and c are bounded. So, since $w_1, w_2 \in \mathrm{SL}(C, X_c)$, they K-fellow travel for some constant K and hence ${}_1w_1v_1$ and ${}_hw_2v_2$ K'-fellow travel for some (larger) constant K'. $\square$

Lemma 6.3. *Let $\widehat{G}$ be a finite group, let V be a finite dimensional torsion-free $\mathbb{Z}\widehat{G}$-module, and W a submodule. Then there exists a $\mathbb{Z}\widehat{G}$-submodule U of V with $U \cap W = \{0\}$ such that $V/(U \oplus W)$ is finite.*

Proof. Let $\widehat{V} = V \otimes \mathbb{Q}$ and $\widehat{W} = W \otimes \mathbb{Q}$ be the corresponding $\mathbb{Q}\widehat{G}$-modules. By Maschke's theorem, there exists a $\mathbb{Q}\widehat{G}$-submodule $\widehat{U}$ of $\widehat{V}$ with $\widehat{V} = \widehat{U} \oplus \widehat{W}$. Let $e_1, \ldots, e_n$ be a $\mathbb{Z}$-basis of V, which we may consider also as a $\mathbb{Q}$-basis of $\widehat{V}$. We can choose a basis $u_1, \ldots, u_k$ of $\widehat{U}$ such that the matrices representing the action of $\widehat{G}$ have integer entries. Define $\lambda_{ij} \in \mathbb{Q}$ by $u_i = \sum_{j=1}^n \lambda_{ij}e_j$. Let m be a common multiple of the denominators of all the λ_{ij}, and define $U \subseteq V$ to be the $\mathbb{Z}$-module generated by the elements $mu_i, \ldots, mu_k$ of $\widehat{U}$. Then $U \oplus W$ has rank n, and so must have finite index in V. $\square$

By Corollary 5.5, the hypotheses of the following result are satisfied in particular when G_1 is a Coxeter group or an Artin group of large type on its natural generating set X_1, and H is a parabolic subgroup.

Proposition 6.4. *Suppose that $H = \langle Y \rangle \le G_1 = \langle X_1 \rangle$ with $Y \subset X_1$, where H and G_1 satisfy the hypotheses of Proposition 5.2 with G_1 and X_1 in place of G and X. Suppose that H is also a subgroup of the finitely generated abelian group G_2. Then $(G_1 *_H G_2, H)$ is strongly synchronously coset automatic.*

Proof. We use the coset language L_1^H constructed in the proof of Proposition 5.2 for (G_1, H) (where this language is SL^H). We extend Y to a generating set X_2 of G_2 such that $H \cap X_2 = Y$ and define the coset language L_2^H for (G_2, H) with respect to X_2 as in the proof of Proposition 6.1, where it is called L^H.

We claim that the language L^H over $X := X_1 \cup X_2$ constructed for $(G_1 *_H G_2, H)$ in the proof of Theorem A is synchronous, so we need to work through that proof in our current context, and we shall adopt the notation used in that proof without further comment. Note first that the graph Λ has two vertices and a single edge, which lies in the maximal tree $\mathcal{T}$, so no letter s_e appears in any deflated words in the language, and Case (4) in the proof of Theorem A does not arise.

Since both L_1^H and L_2^H contain unique representatives of each coset of H, so does L^H. So Case (1) in the proof of Theorem A, where the two different words $w, w' \in L^H$ lie in the same coset, does not arise. But the arguments used in the proof of that case are applied to each of the cases (i.e. cases (2) and (3)) in which $Hwx = Hw'$ with $x \in X$. Recall that $w = u_0u_1 \cdots u_k$, where $e_1e_2 \cdots e_k$ is the path in Λ associated with w. We shall just consider Case (2), in which u_k and x lie in the same subgroup $G_{\tau(e_k)}$. The argument in Case (3) is similar.

In Case (2) we have $u_kx =_G h_ku'_k$ for some $h_k \in H$, where the X-length of h_k, which is the same as its Y-length, is bounded above by some constant K. If $k = 0$, then the synchronous fellow travelling of ${}_1w$ and ${}_{h_0}w'$ follows from the fact that L_1^H and L_2^H are both synchronous automatic coset systems. So we may assume that $k > 0$.

Now, for $1 \le i \le k$, we have $u_{i-1}h_i =_G h_{i-1}u'_{i-1}$ for some $h_{i-1} \in H$. Then $w' = u'_0u'_1 \cdots u'_\ell$ for some $\ell \le k$, where $u'_\ell \ne \epsilon$, and $u_i = \epsilon$ for $\ell < i \le k$. Note that if $h_i =_G 1$ for some i then, by the uniqueness property of L^H, we have $u_j = u'_j$ and $h_j =_G 1$ for all $j < i$.

Suppose next that $|h_i| = 1$ for some i, so h_i is a generator in $Y^{\pm}$. If $u_{i-1} \in L_2^H$, then $h_{i-1} = h_i$ and $u_{i-1} = u'_{i-1}$. If $u_{i-1} \in L_1^H$ then, as stated in the first paragraph of the proof of Proposition 5.2, we have either

(a) $||u_{i-1}| - |u'_{i-1}|| \le 1$ and $h_{i-1} =_G 1$; or
(b) $|u_{i-1}| = |u'_{i-1}|$ and $|h_{i-1}| = 1$.

So in fact one of (a) and (b) must apply irrespective of whether u_{i-1} is in L_1^H or L_2^H.

Now if $|h_i| = m > 1$, then h_i is a product $x_1 \cdots x_m$ of m elements of $Y^{\pm}$. We can then apply the above argument to each of $x_1, \ldots, x_m$ in turn, yielding equations $u_{i-1}x_1 =_G y_1v_1$, $v_1x_2 =_G y_2v_2, \ldots, v_{m-1}x_m =_G y_mv_m$, where each $v_i \in L^H$, each $y_i \in Y^{\pm 1} \cup \{\epsilon\}$, $v_m = u'_{i-1}$, and $h_{i-1} =_G y_1 \cdots y_k$. So, since (a) or (b) applies to each of these equations, we have either

(i) $|h_{i-1}| < |h_i|$ and $||u_{i-1}| - |u'_{i-1}|| \le |h_i|$; or
(ii) $|h_{i-1}| = |h_i|$ and $|u_{i-1}| = |u'_{i-1}|$.

In particular, since $|h_k| \le K$, we have $|h_i| \le K$ for all i. So Case (i) can occur for at most K values of i, and hence

$$\sum_{i=1}^{k} ||u_{i-1}| - |u'_{i-1}|| \le K^2.$$

It is proved in Theorem A that the paths labelled w and w' asynchronously L-fellow travel for some constant L and that, for each i, the beginnings and ends of the subpath labelled u_i correspond to those of u'_i in the fellow travelling. From the above inequality, we see that, if the beginnings of these subpaths labelled u_i and u'_i are at distances i_1 and i_2 from the basepoint, then $|i_1 - i_2| \le K^2$. It follows that w and w' synchronously fellow travel with constant at most $L+K^2$, which completes the proof. $\square$

Proposition 6.5. *Suppose that the group G_1 is finitely generated and hyperbolic relative to a collection of abelian subgroups, and let H be one of those subgroups. Suppose that H is also a subgroup of the finitely generated abelian group G_2. Then $(G_1 *_H G_2, H)$ is strongly synchronously coset automatic.*

Proof. The idea of the proof is first to find coset languages L_1^H and L_2^H for (G_1, H) and (G_2, H) with respect to suitable generating sets X_1 and X_2, then to use Theorem A to find a strong asynchronous automatic coset system L^H for $(G_1 *_H G_2, H)$, and finally to apply Proposition 3.9 to find a synchronous subsystem within L^H. For L_1^H we use the language L_H constructed in the proof of Proposition 4.5, and for L_2^H we use the language also called L_2^H from the proof of Proposition 6.2.

For the application of Proposition 3.9, we need to choose the generating sets X_1 and X_2 for G_1 and G_2 such that $Y := X_1 \cap H = X_2 \cap H$. It is not a problem to find generating sets X_1, X_2 for G satisfying this condition. But the constructions of L_1^H and L_2^H both involve the addition of new generators to Y. We can handle this situation as follows. First we extend X_1 (and so also X_2 and Y) during the construction of L_1^H. Then we further extend X_2 (and so also X_1 and Y) during the construction of L_2^H. Since there is a geodesic biautomatic structure for H on any finite generating set, Lemma 4.4 allows us to reconstruct L_1^H using the new generating set X_1.

We see from the proof of Proposition 4.5 that L_1^H consists of those words in the geodesic biautomatic language L_1 for G_1 that do not begin with a letter in $Y^{\pm}$.

As stated earlier, for the language L_2^H we use the second coset automatic structure in the Proposition 6.2, which is also named L_2^H there. So $X_2 = X_J \cup X_C \cup X_T$, where C and J are disjoint free abelian subgroups of G_2 with $|G_2 : JC|$ finite, $JC \cap H = J$, and X_T contains a transversal for JC in G_2, The elements of L_2^H are words of the form wv, where $w \in \mathrm{SL}(C, X_C)$ and v is a word of length at most 2 over X_T.

Now let $w \in (X^{\pm})^*$ be a shortest representative of its coset of H in G. Then we can write w as $w_1 w_2 \cdots w_k$, where each w_i lies alternately in $(X_1^{\pm})^*$ or in $(X_2^{\pm})^*$, and w_i is a nonempty word that does not begin with a generator from $Y^{\pm}$. We aim to replace w with a word v of the same length in the same coset of H such that, in the corresponding decomposition $v = v_1 v_2 \cdots v_k$, each v_i is in L_1^H or in L_2^H. If we can do this, then $v \in L^H \cap \textsc{Geo}^H$ representing Hw, and we can apply Proposition 3.9 to deduce the existence of a synchronous subsystem of L^H.

Since the words w_i that lie in G_1 must be geodesic words over X_1, we may replace them if necessary by words of the same length representing the same group elements that lie in the geodesic biautomatic language L_1 for G_1. Then, since we are assuming that w_i does not begin with a letter in Y, we have $w_i \in L_1^H$. (This replacement may decrease $|w_i|$ and increase $|w_{i-1}|$, but provided that we replace the words w_i in order of decreasing i, this is not a problem.)

We may assume that the words w_i in G_2 contain no generators in Y, since these could be moved to the left of the word. We may also assume that the letters in w_i from $X_T^{\pm}$ lie at the end of the word. If we have three or more such letters then, from our choice of X_2, we can replace them with a word of the same length containing a generator from Y. So we may replace w_i by a word $v_i = u_1u_2$, where u_1 and u_2 are words over X_C and X_T, respectively, and $|u_2| \leq 2$. We may also assume that u_1 is the shortlex least representative over X_C of the element that it represents, and hence $v_i \in L_2^H$, which completes the proof. $\square$

Example 6.6. In this example we note that there is a pair (G, H) that is strongly synchronously coset automatic, but computer experiments suggest that it does not have λ-limited crossover for any λ with respect to any generating set Y of H. (But we have no means of proving that.) The group G is the trefoil knot group (or the 3-string braid group) $\langle x, y \mid xyx = yxy\rangle$ and H is the free abelian rank 2 subgroup $\langle x, d\rangle$, where d is the central element $(xyx)^2$.

References

[1] J.M. Alonso, Growth functions of amalgams. Arboreal group theory (Berkeley, CA, 1988), 1–34, *Math. Sci. Res. Inst. Publ.*, 19, Springer, New York, 1991.

[2] Y. Antolín, Counting subgraphs in fftp graphs with symmetry, *arXiv:1802.05213*, Preprint, 2018.

[3] Y. Antolín and L. Ciobanu, Finite generating sets of relatively hyperbolic groups and applications to geodesic languages. *Trans. Amer. Math. Soc.*, 368(11):7965–8010, 2016.

[4] G. Baumslag, S.M. Gersten, M. Shapiro and H. Short, Automatic groups and amalgams. *J. Pure Appl. Algebra* 76(3):229–316, 1991.

[5] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[6] N. Brady, Sol geometry groups are not asynchronously automatic. *Proc. London Math. Soc. (3)*, 83(1):93–119, 2001.

[7] B. Brink and R. B. Howlett, A finiteness property and an automatic structure for Coxeter groups. *Math. Ann.*, 296:179–190, 1993.

[8] M. Brittenham, S. Hermiller and T. Susse, The geometry of the word problem for 3–manifold groups. *J. Algebra*, 499:111–150, 2018.

[9] I.M. Chiswell, The growth series of HNN extensions. *Comm. Algebra*, 22(8):2969–2981, 1994.

[10] I.M. Chiswell, The growth series of a graph product. *Bull. London Math. Soc.*, 26(3):268–272, 1994.

[11] L. Ciobanu and S. Hermiller, Conjugacy growth series and languages in groups. *Trans. Amer. Math. Soc.*, 366(5):2803–2825, 2014.

[12] F. Dahmani, Combination of convergence groups. *Geom. Topol.*, 7:933–963, 2003.

[13] P. Dehornoy, The subword reversing method. *Internat. J. Algebra Comput.*, 21(1-2):71–118, 2011.

[14] P. Dehornoy, D.F. Holt and S. Rees, Multifraction reduction IV: Padding and Artin-Tits monoids of sufficiently large type. *J. Pure Appl. Algebra*, 222(12):4082–4098, 2018.

[15] M. Elder, Finiteness and the falsification by fellow traveler property. *Geom. Dedicata*, 95:103–113, 2002.

[16] D.B.A. Epstein, J.W. Cannon, D.F. Holt, S. Levy, M.S. Paterson and W.P. Thurston, *Word Processing in groups.* Jones and Bartlett, 1992.

[17] S. Hermiller and J. Meier, Algorithms and geometry for graph products of groups. *J. Algebra* 171:230–257, 1995.

[18] P.J. Higgins, The fundamental groupoid of a graph of groups. *J. London Math. Soc.* 13(2): 145–149, 1976.

[19] D.F. Holt and D. Hurt, Computing automatic coset systems and subgroup presentations. *J. Symbolic Comput.* 27(1):1–19, 1999.

[20] D.F. Holt, S. Rees and C.E. Röver. *Groups, Languages and Automata.* London Mathematical Society Student Texts, Cambridge University Press, 2017.

[21] D.F. Holt and S. Rees, Artin groups of large type are shortlex automatic with regular geodesics, *Proc. London Math. Soc.*, 104:486–512, 2012.

[22] D.F. Holt and S. Rees, Shortlex automaticity and geodesic regularity in Artin groups, *Groups, Complex. Cryptol.*, 5(1):1–23, 2013.

[23] A. Mann, *How groups grow.* London Mathematical Society Lecture Note Series, 395, Cambridge University Press, Cambridge, 2012.

[24] M. Mihalik, S. Tschantz, Visual decompositions of Coxeter groups. *Groups Geom. Dyn.*, 3(1):173–198, 2009.

[25] Denis V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.

[26] Ian Redfern, Automatic coset systems. PhD thesis, University of Warwick, 1993.

[27] P. Scott and T. Wall, Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge-New York, 1979.

[28] J.-P. Serre, *Trees.* Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[29] M. Shapiro, Automatic structure and graphs of groups. Topology '90 (Columbus, OH, 1990), 355-380, Ohio State Univ. Math. Res. Inst. Publ., 1, de Gruyter, Berlin, 1992.