REWRITING SYSTEMS IN SUFFICIENTLY LARGE ARTIN–TITS GROUPS

EDDY GODELLE AND SARAH REES

ABSTRACT. A conjecture of Dehornoy claims that, given a presentation of an Artin-Tits group, every word that represents the identity can be transformed into the trivial word using the braid relations, together with certain rules (between pairs of words that are not both positive) that can be derived directly from the braid relations, as well as free reduction, but without introducing trivial factors ss^{-1} or $s^{-1}s$. This conjecture is known to be true for Artin-Tits groups of spherical type or of FC type. We prove the conjecture for Artin–Tits groups of sufficiently large type.

1. INTRODUCTION

Let Γ_0 be a labelled finite complete simplicial graph with vertex set S, edge set E and label map $m : E \to \mathbb{N}_{\geq 2} \cup \{\infty\}$. The associated *Coxeter graph* Γ is then obtained from Γ_0 by removing edges labelled by 2, and unlabelling edges whose label is 3. One associates with Γ_0 an *Artin–Tits group* (also known as *Artin group*), denoted by $A(\Gamma_0)$ (or by $A(\Gamma)$), defined by the following *Artin group presentation*:

$$\left\langle\ S \ \middle|\ \underbrace{sts\cdots}_{m} = \underbrace{tst\cdots}_{m}\ ;\ s,t \in S \text{ and } m(\{s,t\}) = m \neq \infty \right\rangle.$$

For instance, if Γ_0 is a complete graph with all edges labelled with ∞, then Γ is equal to Γ_0 and the group $A(\Gamma)$ is the free group on S. Alternatively, if all the labels in Γ_0 are equal to 2, then Γ is completely disconnected and the group $A(\Gamma)$ is the free abelian group on S. More generally, considering graphs Γ_0 whose labels are 2 or ∞ only, one obtains the so-called right-angled Artin groups (RAAG for short), which are the object of numerous articles. Finally, starting from any (undirected) Dynkin diagram and replacing doubled and tripled edges by edges labelled by 4 and 6, respectively, one obtains a Coxeter graph and therefore an Artin group, which is helpful in the study of the associated Coxeter group and the associated Hecke algebra. For instance, if Γ is an unlabelled line and S has cardinality n, then $A(\Gamma)$ is the braid group on $n+1$ strings.

So, the family of Artin groups contains various groups that are of interest. In the present article, we mainly focus on Artin groups where all the labels in Γ_0 are either equal to ∞ or greater or equal to 3. These groups are called *large type Artin groups* and have been considered by various authors [1, 2, 3, 11].

1991 *Mathematics Subject Classification.* 20B30, 20F55, 20F36.

Key words and phrases. Artin–Tits groups, large type, word problem.

Work partially supported by GDRI 571: French-British-German network in Representation theory.

Artin groups are badly understood in general and few results are known. For instance, the word problem is an open problem for Artin groups, although it has been solved for particular subfamilies, including all those introduced above. In general, starting from a finite presentation of a group, there is no general strategy to solve its word problem. From the finite presentation, we can always derive a finite number of elementary transformations on words (that replace some factors of words by other words – see the Preliminary Section 2.1 for the definition of a factor), each of which transforms any word to which it applies into another word with strictly shorter length. And, whenever a sequence of such transformations is applied to an input word w, because the sequence of word lengths is strictly decreasing, this process must stop after at most $|w|$ such elementary transformations. Where the group presentation is hyperbolic, the resulting word is the empty word if and only if the initial word represents the identity element of the group. So, using the bound on the number of possible successive elementary transformations, one gets a solution to the word problem. But, when the group presentation is not hyperbolic, there must be representatives of the identity element that are not reduced by the finite set of transformations to the empty word, and hence the process does not solve the word problem.

Unfortunately, except in the case of free groups, Artin groups are not hyperbolic as they contains copies of $\mathbb{Z}^2$, so the approach above does not work to solve the word problem in Artin groups. However, one might expect that an alternative family of elementary transformations could lead to a solution to the word problem for any Artin group. It is natural to look for a solution with a set of elementary transformations on words that do not increase length. For instance, the solution to the word problem obtained in [15] for right-angled Artin groups can be seen as a special example of this strategy.

Definition 1.1. For a finite presentation $\langle S, R\rangle$, we define *special transformations* on finite words on $S \cup S^{-1}$ as follows:
- *type* 0 (*resp.* ∞): Remove (*resp.* insert) some pair $s^{-1}s$ or ss^{-1}, where s is a letter of S,
- *type* 1: Replace some factor v by u, or v^{-1} by u^{-1}, where $u = v$ is a relation of R.

Since the presentation is finite, the set of special transformations is finite. But note that, throughout the paper, we shall abuse notation and consider relations $u = v$ and $v = u$ to be the same relation, and hence, in particular, we shall also consider that there are four (rather than two) type 1 transformations associated with each relation $u = v$.

From now on, for a set X of types of transformations on words, we write $w \overset{X}{\rightsquigarrow}_R w'$ if w can be transformed to w' using transformations of type X. Thus, for a group G with finite presentation $\langle S, R\rangle$ and for every word w on $S \cup S^{-1}$, we have

$$w \overset{0,1,\infty}{\rightsquigarrow}_R \varepsilon \quad \Longleftrightarrow \quad \overline{w} = 1, \tag{1.1}$$

where ε is the empty word and $\overline{w}$ is the image of w in the group G. Of course here we do not get a solution to the word problem as long as we do not have a bound on the length of the path from w to ε using relations of types $0, 1, \infty$ — such a bound does not exist in general. If one wants to obtain a set of elementary transformations that do not increase the length then it is legitimate to include transformations of type 0 or 1, whereas transformations of type ∞ have to be avoided. Clearly if $w \overset{0,1}{\rightsquigarrow}_R \varepsilon$ then $\overline{w} = 1$ but the converse is not true in general, nor in the special case of Artin group presentations (see [8] for an example). It is therefore natural to address the question of whether one may add some other special transformations to obtain an equivalence for all Artin groups. Two partial results on that direction appear recently in the literature [8, 11].

In [11] a family of elementary operations, called *of type* β, is introduced. A special case of the main result proved in [11] is that for any Artin group presentation $\langle S, R\rangle$ of large type and for every word w on $S \cup S^{-1}$, we have

$$w \overset{0,1,\beta}{\rightsquigarrow}_R \varepsilon \quad\iff\quad \overline{w} = 1. \tag{1.2}$$

The type β operations do not increase the length, and so this leads to a solution of the word problem. However type β operations are not finite in number (even if for a given length only a finite number of them can be applied) and given a word, it is not easy to detect where a type β transformation can be applied. Moreover, as shown in [11], one cannot expect that the above equivalence holds for every Artin group.

Now comes the crucial point. In [8], some special transformations of *type* 2 are introduced, falling into two types called *type* $2r$ ("type 2-right") and *type* $2l$ ("type 2-left), which allow some limited substitution using unsigned words. These are defined as follows:

Definition 1.2. Let $G = \langle S, R\rangle$ be a group, and suppose that $u = v$ is a relation in R, between two positive words u, v. Suppose that u can be written as a concatenation u_1u_2 and v as a concatenation v_1v_2. Suppose that w is a word over $S \cup S^{-1}$. We define a transformation that replaces a factor $v_1^{-1}u_1$ of w by $v_2u_2^{-1}$ to be of *type* $2r$ ("type-2-right"), provided that both v_1 and u_1 are non-empty. We define a transformation that replaces a factor $v_2u_2^{-1}$ of w by $v_1^{-1}u_1$ to be of *type* $2l$ ("type-2-left"), provided that both v_2 and u_2 are non-empty.

Basically, the rules of type 2 are those directly derived from braid relations for which the left hand side of the rule is unsigned. Note that in the special case where all four of u_1, u_2, v_1, v_2 are non-empty, the associated transformations of types $2l$ and $2r$ are inverse to each other.

Example 1.3. Consider a *dihedral* Artin group with $S = \{s, t\}$ and $m_{s,t} = 3$. Then the transformations

$$s^{-1}t^{-1}s \to ts^{-1}t^{-1} \text{ and } t^{-1}s^{-1}t^{-1}s \to s^{-1}t^{-1}$$

are both of type $2r$, the first one with $u_1 = s$, $u_2 = ts$, $v_1 = ts$ and $v_2 = t$, and the second one with $u_1 = s$, $u_2 = ts$,$v_1 = tst$ and v_2 empty. Then, the reverse of the first,

$$ts^{-1}t^{-1} \to s^{-1}t^{-1}s$$

is type $2l$, but the reverse of the second,

$$s^{-1}t^{-1} \to t^{-1}s^{-1}t^{-1}s$$

is not of type $2l$.

It is immediate that the set of transformations of type 2 is finite, and therefore so is the set of transformations of types $0, 1$ or 2. Moreover, it is clear that $w \overset{0,1,2}{\rightsquigarrow}_R \varepsilon$ implies $\overline{w} = 1$. The reader may wonder why operations of type 2 should be considered of special interest. The explanation comes from the case of *spherical* *type* Artin presentations (that is, those based on finite type Dynkin diagrams, and so in particular all braid groups).

In this case, $w \overset{0,1}{\rightsquigarrow}_R \varepsilon \iff \overline{w} = 1$ does not hold,but $w \overset{0,2}{\rightsquigarrow}_R \varepsilon \iff \overline{w} = 1$ does [8]. Further, in such a group, a sequence of operations of types $2r$ (resp. $2l$) and 0 allows one to transform any (unsigned) word w into a word of the form $w_1 w_2^{-1}$ (resp. $w_1^{-1} w_2$) for which w_1, w_2 are positive words with no common right (resp. left) divisor in the associated Artin-Tits monoid. (See [8] and Propositions 2.6 and 2.8 below.)

The result we are going to prove here is

Theorem 1.4. *For every sufficiently large Artin-Tits presentation $\langle S, R\rangle$, and any word $w \in S \cup S^{-1}$,*

$$w \overset{0,1,2}{\rightsquigarrow}_R \varepsilon \iff \overline{w} = 1 \tag{1.3}$$

Our definition of sufficiently large type for an Artin-Tits presentation is taken from [12], but we repeat it as Definition 3.1; note that this definition includes all presentations of large type.

It is conjectured in [9, Conj. 3.28] that the equivalence of Theorem 1.4 holds for all Artin-Tits presentations. The conjecture is repeated in [8, Conj. 1.6], where it is proved for Artin-Tits presentations of spherical and FC types.

Definition 1.5. A finite presentation $\langle S, R\rangle$ of a group G satisfies Property H if the equivalence $w \overset{0,1,2}{\rightsquigarrow}_R \varepsilon \iff \overline{w} = 1$ holds for every word w on $S \cup S^{-1}$.

So, we can rephrase [8, Conj. 1.6] as a claim that Property H is satisfied by every Artin presentation, and Theorem 1.4 as a verification of that claim for any Artin presentation of sufficiently large type. The conjecture was already proved in [8] for spherical type Artin groups, and then more generally for all Artin groups of FC type, and so all right-angled Artin groups. But most Artin groups of sufficiently large type (and in particular those of large type) are not of FC type, and so are not covered by the results of [8]; so certainly our result provides a new family of Artin groups for which the conjecture is valid. We prove the result for groups of sufficiently large type as an extension of the result for groups of large type, recognising that the first of those two classes of groups can be built out of the second in the same way that the groups of FC type can be built out of the groups of spherical type; so this part of our proof follows the proof of [8]. In order to obtain their result for Artin groups of FC type, the authors of [8] proved that a stronger but more technical property, namely Property $H^{\#}$ (defined in [8]), is satisfied by Artin groups of spherical type. We will prove that this property holds for Artin groups of large type. Then, combining this result with [8, Prop. 2.14] we will derive the following.

Theorem 1.6. *Let $\mathcal{T}$ be the family of those Artin-Tits presentations (S, R) for which every connected full subgraph of the Coxeter graph Γ without an ∞-labelled edge is either of spherical type or of large type. Then every presentation (S, R) in $\mathcal{T}$ satisfies Property $H^{\#}$ and (therefore) Property H. Hence, for every word w on $S \cup S^{-1}$,*

$$w \overset{0,1,2}{\rightsquigarrow}_R \varepsilon \iff \overline{w} = 1.$$

The reader should note that transformations of type 2 can increase the length of words and therefore our result does not lead directly to a solution to the word problem. However, whenever we can prove a bound on the maximal number of iterations of special transformations we can derive a solution to the word problem. We might expect that such a bound could be found in all cases, even if a general argument for that remains to be found. The main argument of our proof will consist of a proof that that any transformation of type β can be decomposed as a sequence of type 2 and type 1 transformations. The remainder of this article is organised as follows. Section 2 is devoted to the proof of Property H for Artin-Tits groups of large type, stated at the end of the section as Corollary 2.21. Section 3 considers Artin-Tits groups of sufficiently large type. Those groups are introduced at the beginning of Section 3, and characterized in Section 3.1 as a particular class of Artin groups closed under amalgamation over standard parabolic subgroups. Property $H^{\#}$ is defined in Section 3.3, and Proposition 3.5 [8, Proposition 2.14] is stated, which will be used to extend the proof of Property $H^{\#}$ for large type groups to a larger class of Artin-Tits groups, which (according to the given characterization) must include all those of sufficiently large type. Section 3.4 introduces the machinery of critical factorizations and critical sequences of transformations for large type groups from [11]. Section 3.5 is devoted to the proof that Artin groups of large type satisfy Property $H^{\#}$, and the final section, Section 3.6, to the proofs of Theorems 1.6 and 1.4.

We would like to acknowledge some helpful comments from the referee, which led us to introduce property H^{+}, strengthen Corollary 2.21 as Corollary 2.24, and prove Proposition 3.3.

2. Property H For Artin-Tits groups of large type

In this section we prove that Property H holds for Artin-Tits group of large type. Throughout this section, we fix a Coxeter graph Γ. We denote by Γ_0 the associated labelled full simplicial graph, as defined in the introduction. We assume that the associated Artin-Tits group is of large type; recall that this means that any edge label is either infinite or at least 3. In order to prove the result, we start with the crucial notion of a *critical word* introduced in [11]. As before, by S we denote the set of vertices of Γ

2.1. **Preliminaries.**

We start with some terminology. Let S be a finite set. We denote by $(S \cup S^{-1})^*$ the set of words on $S \cup S^{-1}$. A word w on $S \cup S^{-1}$ will be called *positive* if it is written over S. It will be called *negative* if it is written over S^{-1}. A *signed* word is a word that is either positive or negative. Some words are neither positive nor negative; when this is the case, we say that the word w is *unsigned*. We denote by w^{-1} the word that is the formal inverse of w. If G is a group generated by S, a element g of G will be called *positive* (relative to S) when it possesses a positive representative word over S.

A word v on $S \cup S^{-1}$ will be called a *factor* of a word w if the word w can be decomposed with $w = w_1 v w_2$ for some words w_1, w_2.

A word w is called *freely reduced* if it does not contain a factor of the form either ss^{-1} or $s^{-1}s$ for some s in S. Equivalently, w is freely reduced if no operation of type 0 can be applied.

When x and y are two words and m is a positive integer, we denote by $[x, y, j\rangle$ the word $xyxy\cdots$ obtained as the alternating concatenation of the words x and y that concatenates j words, the first being x. Hence $[x, y, 5\rangle = x[y, x, 4\rangle = xyxyx$. Similarly we denote by $\langle y, x, j]$ the word $\cdots yxyx$ obtained as the alternating concatenation of the words x and y, that concatenates j words, the last being x. So

$\langle y, x, 6] = yxyxyx$. In particular $[x, y, j\rangle = \langle y, x, j]$ or $[x, y, j\rangle = \langle x, y, j]$ depending on whether j is odd or even. For instance, $[x, y, 5\rangle = xyxyx = \langle y, x, 5]$ but $[x, y, 4\rangle = xyxy = \langle x, y, 4]$.

Finally, we say that a positive word is *square-free* if it does not contain a factor of the form ss for any s in S. Now, for a word w, we denote by $\tilde{p}(w)$ the length of the longest square-free positive factor of w. Similarly we denote by $\tilde{n}(w)$ the length of the longest negative factor of w whose formal inverse is square-free.

2.2. **Dihedral Artin-Tits groups.** In the case where Γ has only 2 vertices s, t, the associated Artin group $A(\Gamma_0)$ is called a *dihedral* Artin group. Indeed, in the case where $m = m(\{s, t\}) \neq \infty$, if one adds the relations $s^2 = 1$ and $t^2 = 1$ to the presentation of $A(\Gamma_0)$, then one obtains the associated Coxeter group. On the other hand the obtained presentation is the one of a dihedral group of order $2m$. In particular the latter group is finite. Following [13], for a word w we set $p(w) = \min(\tilde{p}(w), m)$ and $n(w) = \min(\tilde{n}(w), m)$.

In order to state the next result, let us introduce a restricted version of the operation of type 2.

Definition 2.1. For a finite presentation of a group $\langle S, R\rangle$, we define *special transformations* of type $2^\star$ on words on $S \cup S^{-1}$ to be special transformations of type 2 for which, following the notation of Definition 1.2,
- in type $2r$ with $w = w_1 v^{-1} v' w_2$, the word v is the greatest left common factor of the words vu and vw_1^{-1} and the word v' is the greatest left common factor of the words $v'u'$ and $v'w_2$;
- in type 2ℓ with $w = w_1 v v'^{-1} w_2$, the word v is the greatest right common factor of the words uv and $w_1 v$ and the word v' is the greatest right common factor of the words $u'v'$ and $w_2^{-1}v'$.

Example 2.2. Consider a *dihedral* Artin group with $S = \{s, t\}$ and $m_{s,t} = 4$. Consider the word $t^2sts^{-1}t$. Then, we have $t^2\mathbf{sts^{-1}}t \overset{2}{\rightsquigarrow}_R t^2\mathbf{t^{-1}s^{-1}tst}t$, but this transformation is not of type $2^\star$ because $v = st$; $uv = stst$ and $w_1 v = t^{-1}tst$. So the word tst is a greater common right divisor of uv and $w_1 v$ than v. However, $t\mathbf{tsts^{-1}}t \overset{2^\star}{\rightsquigarrow}_R t\mathbf{s^{-1}tst}t$. The reader should note that in both cases the last letter of w_1 is t, but in the first case tv is square-free, whereas in the second case tv is not.

For the remainder of this section, we assume that Γ_0 has only 2 vertices s, t (so $S = \{s, t\}$), and set $m = m_{s,t}$. So the group $A(\Gamma_0)$ is a dihedral Artin group. We set

$$\Omega(\Gamma_0) = \big\{w \in (S \cup S^{-1})^* \mid w \text{ is freely reduced with} \max(p(w), n(w)) < m\big\}$$

The reader should note that the definition of $\Omega(\Gamma_0)$ ensures that operations of types 0 or 1 cannot be applied to words in $\Omega(\Gamma_0)$; only operations of type 2 can apply. Moreover, if w belongs to $\Omega(\Gamma_0)$, then $\tilde{p}(w) = p(w)$ and $\tilde{n}(w) = n(w)$.

For a non-empty word w in $\Omega(\Gamma_0)$, we define the *$\star$-decomposition of* w to be the unique sequence $(v_1, \ldots, v_k)$ of non-trivial words such that $w = v_1 \cdots v_k$ and each factor v_i is the maximal square-free signed left factor of the word $v_i \cdots v_k$.

Proposition 2.3. *Let w be a non-empty word that belongs to $\Omega(\Gamma_0)$, and denote by $(v_1, \cdots, v_k)$ the $\star$-decomposition of w. Assume that the word w' is obtained from w by an operation of type $2^\star$. Then, w' belongs to $\Omega(\Gamma_0)$ too. Moreover, there exists a* *subscript i such that the $\star$-decomposition of w' is $(v_1, \cdots, v_{i-1}, u_i, u_{i+1}, v_{i+2}, \cdots, v_k)$ and the $2^\star$-operation transforms w into w' by replacing the factor v_iv_{i+1} of w by the word u_iu_{i+1}.*

Proof. Since the Artin group is assumed dihedral, the set R consists of a single relation $[s,t,m\rangle = [t,s,m\rangle$. Write $w = w_1w_2w_3$ and $w' = w_1w_2'w_3$, where w_2' is the factor of w' obtained from the factor w_2 of w by applying an operation of type $2^\star$. We have either $w_2 = v^{-1}v'$ or $w_2 = vv'^{-1}$. In the first case, $w_2' = uu'^{-1}$ where $vu = v'u'$ is the unique relation in R. In the second case, $w_2' = u^{-1}u'$ where $uv = u'v'$ is the unique relation in R.

Assume the first case. The arguments for the second case are similar and left to the reader. We can assume without restriction that $v = [s,t,j\rangle$ and $v' = [t,s,\ell\rangle$ for some $j, \ell < m$. Then $vu = [s,t,m\rangle$ and $v'u' = [t,s,m\rangle$. Since the words $v_1, \cdots, v_k$ are signed words and any word v_iv_{i+1} is either unsigned or not square-free, there has to exist a subscript i so that v^{-1} is a right factor of v_i and v' is a left factor of v_{i+1}. Now the operation that transforms w into w' is not just of type 2 but actually of type $2^\star$. This imposes $v^{-1} = v_i$ and $v' = v_{i+1}$. For a in $S = \{s,t\}$, denote by $\hat{a}$ the other letter of S. Denote by a and a' the last letter of v and v', respectively. Then, the first letter of u is $\hat{a}$ and the last letter of u'^{-1} is $(\hat{a'})^{-1}$. Denote by c the last letter c of w_1, that is, the last letter of v_{i-1}, and by d the first letter of w_3, that is, the first letter of v_{i+2}.

The only cases where w' could not be freely reduced would be if either $c = \hat{a}^{-1}$ or $d = \hat{a'}$. If c were equal to $\hat{a}^{-1}$ then we would have $vc^{-1} = [s,t,j+1\rangle$ with $j+1 \leq n(w) < m$. so, vc^{-1}, that is, $v\hat{a}$, would be a common left factor of vw_1^{-1} and vu, contradicting the fact that the transformation is of type $2^\star$. So $c \neq \hat{a}^{-1}$. Similarly, $d \neq \hat{a'}$. Hence, w' is a freely reduced word.

We prove now that $\max(p(w'), n(w')) < n$. This will imply that w' belongs to $\Omega(\Gamma_0)$. Assume that one of the words $[s,t,m\rangle$ and $[s,t,m\rangle$ or their inverses is a factor of w'. Since w belongs to $\Omega(\Gamma_0)$, this word cannot be a factor of w_1 or of w_3. On the other hand, it can not be a factor of w_2'. Then that word has to either overlap w_1 and w_2', or to overlap w_2' and w_3. In the first case, it has to contain the word $c\hat{a}$ as a factor, which implies that $c = a$. In the second case it has to contain the word $d^{-1}\hat{a'}$ which implies that $d = a'^{-1}$. Either of these would contradict that fact that w is freely reduced. We deduce that $\max(p(w'), n(w')) < m$, and hence w' belongs to $\Omega(\Gamma_0)$.

From $c \neq a$ and $d \neq a'^{-1}$, we further deduce that neither $v_{i-1}\hat{a}$ nor $u'^{-1}d$ can be a square-free signed word. Hence $(v_1, \cdots, v_{i-1}, u, u'^{-1}, v_{i+2}, \cdots, v_k)$ is the $\star$-decomposition of w'. ☐

From Proposition 2.3 we deduce that

Corollary 2.4. *Assume that $A(\Gamma_0)$ is a dihedral Artin group. Then, $\Omega(\Gamma_0)$ is closed under the operations of type $2^\star$. Moreover, if w, w' belong to $\Omega(\Gamma_0)$ then*

$$w \overset{2^\star}{\rightsquigarrow}_R w' \iff w' \overset{2^\star}{\rightsquigarrow}_R w.$$

2.3. **Normal decomposition.** In the previous section we have seen that, in the case of a dihedral Artin-Tits group, the set $\Omega(\Gamma_0)$ is closed under operations of type $2^\star$. Here we study the connected components of $\Omega(\Gamma_0)$ under the $2^\star$ operation.

Proposition 2.5. *Assume that $A(\Gamma_0)$ is a dihedral Artin-Tits group. Assume that w and w' belong to $\Omega(\Gamma_0)$, then,*

$$w \overset{2^\star}{\rightsquigarrow}_R w' \iff w \text{ and } w' \text{ represent the same element in } A(\Gamma_0).$$

To prove this result we use the fact that dihedral Artin-Tits groups are contained in the more general family of spherical type Artin-Tits groups. We recall that an Artin-Tits group is of spherical type when its associated Coxeter group is finite, which is the case for dihedral Artin-Tits groups with $m < \infty$ as explained in the previous section. The crucial argument is the following.

Proposition 2.6. [5] *Let $A(\Gamma_0)$ be a spherical type Artin group, and let $g \in A(\Gamma_0)$. Then there exists a unique pair of positive elements (g_1, g_2) such that (1) $g = g_1 g_2^{-1}$ and (2) whenever h_1, h_2 are positive elements with $g = h_1 h_2^{-1}$, then there exists a positive element h with $h_1 = g_1 h$ and $h_2 = g_2 h$.*

The above decomposition $g_1 g_2^{-1}$ is called the left normal decomposition of g. The following is an immediate consequence of the above proposition.

Proposition 2.7. *Assume that $A(\Gamma_0)$ is a spherical type Artin group. Let g belong to $A(\Gamma_0)$. Assume that (w_1, w_2) is a pair of positive words so that $w_1 w_2^{-1}$ is a word representative of g. Denote by g_1 and g_2 the (positive) elements of $A(\Gamma_0)$ represented by w_1 and w_2 respectively. Then, the following are equivalent:*

(1) *$g_1 g_2^{-1}$ is not the left normal decomposition of g,*
(2) *there exist two distinct words w and w' so that $w_1 w_2^{-1} \overset{1}{\rightsquigarrow}_R w \overset{0}{\rightsquigarrow}_R w'$.*

Using this property, we can deduce the following.

Proposition 2.8. *Let $A(\Gamma_0)$ be a dihedral Artin-Tits group, and suppose that $w \in \Omega(\Gamma_0)$. Denote by g the element of $A(\Gamma_0)$ represented by g and by (g_1, g_2) the pair of positive elements of $A(\Gamma_0)$ corresponding to the left normal decomposition of g.*

Then, there exists a pair of positive words (w_1, w_2) so that the words w_1 and w_2 represent g_1 and g_2, respectively, and

$$w \overset{2^\star}{\rightsquigarrow}_R w_1 w_2^{-1}.$$

Proof. Let $(v_1, \dots, , v_k)$ be the $\star$-decomposition of w. Say that the pair (i, j), with $i < j$, is an inversion of the word w if v_i is a negative word whereas v_j is a positive word. Denote by $\mathrm{Inv}(w)$ the number of inversions of the word w.

Assume first $\mathrm{Inv}(w) = 0$. Then there exists an index i such that $v_1 \cdots v_i$ is a positive (or empty) word and $v_{i+1} \cdots v_k$ is negative (or empty). Since $w \in \Omega(\Gamma_0)$, it follows (immediately from the definition of $\Omega(\Gamma_0)$) that no operation of type 1 or 0 can be applied to w. Hence, we deduce from Proposition 2.7 that the words $v_1 \cdots v_i$ and $(v_{i+1} \cdots v_k)^{-1}$ represent g_1 and g_2, respectively.

So now suppose that $\mathrm{Inv}(w) \neq 0$. Then there exists an index i such that v_i is a negative word whereas v_{i+1} is positive. By definition of the decomposition $v_1 \cdots v_k$, an operation of type $2^\star$ can be applied to w to replace the word $v_i v_{i+1}$ by the word $u_i u_{i+1}$ for which $v_{i+1} u_{i+1}^{-1} = v_i^{-1} u_i$ is the unique relation in the presentation of the group $A(\Gamma_0)$. We obtain the word $w' = v_1 \cdots v_{i-1} u_i u_{i+1} v_{i+2} \cdots v_k$. By Proposition 2.3, the word w' remains in $\Omega(\Gamma_0)$ and $(v_1, \dots, v_{i-1}, u_i, u_{i+1}, v_{i+2}, \dots, v_k)$ is the $\star$-decomposition of w'. It is immediate that $\mathrm{Inv}(w') = \mathrm{Inv}(w) - 1$. Repeating the argument, we deduce there exists a word w'' in $\Omega(\Gamma_0)$ such that $w \overset{2^\star}{\rightsquigarrow}_R w''$ and $\mathrm{Inv}(w'') = 0$. Denote by $(u_1, \dots, u_k)$ the $\star$-decomposition of the obtained word

w''. By the first case, there exists an index i such that the words $u_1 \cdots u_i$ and $(u_{i+1} \cdots u_k)^{-1}$ represent g_1 and g_2, respectively. This proves the proposition. □

We can now prove Proposition 2.5.

Proof of Proposition 2.5. Assume that w and w' belong to $\Omega(\Gamma_0)$.

Certainly if $w \overset{2\star}{\rightsquigarrow}_R w'$, then w and w' represent the same element in $A(\Gamma_0)$.

Conversely, assume w and w' represent the same element g, and denote its left normal decomposition by $g_1 g_2^{-1}$. By Proposition 2.8, there exist positive words w_1, w_1', w_2, w_2' so that (a) w_1 and w_1' represent g_1; (b) w_2 and w_2' represent g_2; (c) $w \overset{2\star}{\rightsquigarrow}_R w_1 w_2^{-1}$ and $w' \overset{2\star}{\rightsquigarrow}_R w_1' {w'}_2^{-1}$.

From (c) and Corollary 2.4 we have $w_1' {w'}_2^{-1} \overset{2\star}{\rightsquigarrow}_R w'$. By (a), (b) and [14], we have $w_1 \overset{1}{\rightsquigarrow}_R w_1'$ and $w_2 \overset{1}{\rightsquigarrow}_R w_2'$. Since w and w' belong to $\Omega(\Gamma_0)$, the words $w_1 w_2^{-1}$ and $w_1 w_2^{-1}$ must too, by Corollary 2.4, and hence no non-trivial operations of type 1 can be applied to them or to their factors. Hence, $w_1 = w_1'$ and $w_2 = w_2'$. It follows that $w \overset{2\star}{\rightsquigarrow}_R w_1 w_2^{-1} \overset{2\star}{\rightsquigarrow}_R w'$. □

2.4. **Critical words.** For all of this section, we assume that $A(\Gamma_0)$ is a dihedral Artin group with $S = \{s, t\}$ and $m = m_{s,t} \neq \infty$. Set $\Delta = \overline{[s, t, m\rangle} = \overline{[t, s, m\rangle}$. Then, $\Delta \overline{s}^{\pm 1} = \overline{\delta(s)}^{\pm 1} \Delta$ and $\Delta \overline{t}^{\pm 1} = \overline{\delta(t)}^{\pm 1} \Delta$, where δ is the permutation of S defined by $\delta(s) = \overline{s^\Delta}, \delta(t) = \overline{t^\Delta}$, acting as the identity when m is even, and swapping s and t when m is odd. The map δ naturally extends to a involution δ on $(S \cup S^{-1})^*$ such that for every word v on $(S \cup S^{-1})^*$ one has $\Delta \overline{v} = \overline{\delta(v)} \Delta$.

We recall that for a group G generated by a set S, a *geodesic word representative* of an element g is a representative word of g over $S \cup S^{-1}$ that is of minimal length amongst the representative words of g. Given an arbitrary Artin-Tits group, it is an open question to obtain an algorithm that for each word returns a geodesic word representing the same element. However, in the case of dihedral Artin-Tits groups, and in the more general case of Artin-Tits groups of large type, a positive answer has been obtained in [13] and in [11], respectively.

More precisely, it is proved in [13], in a dihedral group A_S, a freely reduced word w over s, t is geodesic if and only if $p(w) + n(w) \leq m$. In addition in [11, Section 2], the authors introduced the notion of *critical word*. A critical word may be signed or unsigned; for now we restrict to unsigned critical words.

Definition 2.9. Assume $A(\Gamma_0)$ is a dihedral Artin group. A freely reduced unsigned geodesic word w with $p(w) + q(w) = m$ is called an *unsigned critical word* if either

$$w = [x, y, p\rangle \; w' \; [z, t, q\rangle^{-1}$$

or

$$w = [x, y, q\rangle^{-1} \; w' \; [z, t, p\rangle$$

with $\{x, y\} = \{z, t\} = S$, $p = p(w)$ and $q = q(w)$.

Obviously the conditions $p(w) = p, n(w) = n$ impose some restrictions on the word w'.

Recall that $[x, y, p\rangle = \langle x', y', p]$ where $\{x, y\} = \{x', y'\}$, and that $[z, t, q\rangle = \langle z', t', q]$ where $\{z, t\} = \{z', t'\}$. Now, let $w = [x, y, p\rangle \; w' \; [z, t, q\rangle^{-1}$ be a word over $(S \cup S^{-1})$ such that $p + q = m$. One has

$$\overline{w} = \overline{[x, y, p\rangle \, w' \, [z, t, q\rangle^{-1}} = \overline{\langle x, y, q]^{-1} \; \Delta \, w' \, [z, t, q\rangle^{-1}} = \overline{\langle x, y, q]^{-1} \; \delta(w') \Delta \, [z, t, q\rangle^{-1}}$$

$$= \overline{\langle x,y,q]^{-1}\,\delta(w')\,\langle z,t,p]}$$

Set

$$\tau\left([x,y,p\rangle\; w'\; [z,t,q\rangle^{-1}\right) = \langle x,y,q]^{-1}\,\delta(w')\,\langle z,t,p];$$

then one has $\overline{\tau(w)} = \overline{w}$. So, w is unsigned and geodesic if and only if $\tau(w)$ is. Moreover, w is an unsigned critical word with $p = p(w)$ and $q = q(w)$ if and only if $\tau(w)$ is an unsigned critical word with $p = p(\tau(w))$ and $q = q(\tau(w))$. In this case, we also set

$$\tau\left(\langle x,y,q]^{-1}\,\delta(w')\,\langle z,t,p]\right) = [x,y,p\rangle\; w'\; [z,t,q\rangle^{-1}.$$

Then τ is an involution over the set of unsigned critical words.

The above is proved, together with other properties of critical words in [11, Proposition 2.1]. From those properties we shall also need the following, in Section 3.5:

Lemma 2.10. *Assume that $A(\Gamma_0)$ is a dihedral Artin group with generating set $S = \{s,t\}$. Let w be an unsigned critical word. The rightmost letter of w is $s^{\pm 1}$ if and only if the rightmost letter of $\tau(w)$ is $t^{\mp 1}$.*

Lemma 2.11. *Assume $A(\Gamma_0)$ is a dihedral Artin group with generating set $S = \{s,t\}$. Let w be an unsigned critical word. Then $w \overset{2\star}{\rightsquigarrow}_R \tau(w)$.*

Proof. The words w and $\tau(w)$ are unsigned critical words. In particular $p(w) + q(w) = m$ with $p(w) \neq 0$ and $q(w) \neq 0$. So w is a freely reduced word so that $p(w) < m$ and $q(w) < m$, and hence w belongs to $\Omega(\Gamma_0)$. But $\tau(w)$ is also a reduced word with $p(\tau(w)) = p(w) < m$ and $q(\tau(w)) = q(w) < m$, and hence also $\tau(w)$ belongs to $\Omega(\Gamma_0)$, too. Since $\overline{\tau(w)} = \overline{w}$, Proposition 2.5 proves that $w \overset{2\star}{\rightsquigarrow}_R \tau(w)$. ◻

In [11], the authors also introduced a notion of a positive (*resp.* negative) critical word. If $A(\Gamma_0)$ is a dihedral Artin group with generating set $S = \{s,t\}$, a positive word w with $\tilde{p}(w) = p(w) = m$ is called critical if it has the form $[x,y,m\rangle w'$ or $w'\langle m,x,y]$ with $\{x,y\} = \{s,t\}$ and $p(w') < m$. Similarly a negative word w with $\tilde{n}(w) = n(w) = m$ is called critical if it has the form form $[x^{-1},y^{-1},m\rangle w'$ or $w'\langle m,x^{-1},y^{-1}]$, with $\{x,y\} = \{s,t\}$ and $n(w') < m$. Further, the definition of τ can be extended to that of an involution on critical words that maps each positive, resp. negative critical word to another positive resp. negative critical word representing the same group element. More precisely, if $w = [x,y,m\rangle w'$ is a positive critical word, then $\overline{[x,y,m\rangle w'} = \overline{\delta(w')\langle x,y,m]} = \overline{\delta(w')\langle y,x,m]}$. Only one of the two words $\delta(w')\langle x,y,m]$ and $\delta(w')\langle y,x,m]$ has different last letter from w. If w' is not empty it is also the only one of these two words that is critical. We define $\tau(w)$ to be this word. We define $\tau(w)$ for a negative critical word $w = w'\langle x,y,m]$ in a similar way. It is easy to see that τ remains an involution.

It is straightforward to prove the following, and so we omit the details:

Lemma 2.12. *Assume that $A(\Gamma_0)$ is a dihedral Artin group with generating set $S = \{s,t\}$. For any positive or negative critical word w, $w \overset{1}{\rightsquigarrow}_R \tau(w)$.*

We now define the special transformation β in the case of dihedral Artin-Tits groups:

Definition 2.13. Assume that $A(\Gamma_0)$ is a dihedral Artin group with generating set $S = \{s, t\}$. Let τ be the above involution on the set of critical words. We define *special transformations* of type β on words on $S \cup S^{-1}$ as follows:
type β : Replace some factor w with $\tau(w)$, where w is a critical word.

From this definition and Lemmas 2.11 and 2.12, it follows:

Lemma 2.14. *Assume that $A(\Gamma_0)$ is a dihedral Artin group with generating set $S = \{s, t\}$. For any two words w, w', one has*

$$w \overset{\beta}{\rightsquigarrow}_R w' \implies w \overset{1,2^\star}{\rightsquigarrow}_R w'.$$

Proposition 2.15. [11, Sec. 2] *Assume that $A(\Gamma_0)$ is a dihedral Artin group. Let w be a word over $S \cup S^{-1}$. Then there exists a geodesic word w' such that $\overline{w} = \overline{w'}$ and*

$$w \overset{0,\beta}{\rightsquigarrow}_R w'.$$

Proof. The result follows immediately from [11, Lemma 2.3], from which we deduce that if w is freely reduced but non-geodesic then it must possess a critical word w_1 as a factor, and then the word w' obtained from w by replacing the factor w_1 with $\tau(w_1)$ is not freely reduced. So a combination of free reduction and at most one type β transformation reduces w to w'. □

Combining Lemma 2.14 and Proposition 2.15 we get the following corollary.

Corollary 2.16. *Assume that $A(\Gamma_0)$ is a dihedral Artin group with S as generating set. Let w be a word over $S \cup S^{-1}$. There exists a geodesic word w' such that $\overline{w} = \overline{w'}$ and*

$$w \overset{0,1,2^\star}{\rightsquigarrow}_R w'.$$

Transformations of type $2^\star$ are special cases of transformations of type 2 and the unique geodesic representative word of 1 is the empty word ε. Therefore as a consequence of the above corollary, we recover :

Corollary 2.17. [8] *Assume $A(\Gamma_0)$ is a dihedral Artin group with S as generating set. Let w be a word over $S \cup S^{-1}$. Then,*

$$w \overset{0,1,2}{\rightsquigarrow}_R \varepsilon \iff \overline{w} = 1$$

2.5. **The case of large type Artin-Tits groups.** Our objective here is to prove the following result, which is a special case of Theorem 1.4.

Proposition 2.18. *Assume $A(\Gamma_0)$ is an Artin group of large type with S as generating set. Let w be a word over $S \cup S^{-1}$. There exists a geodesic word w' such that $\overline{w} = \overline{w'}$ and*

$$w \overset{0,1,2^\star}{\rightsquigarrow}_R w'.$$

The proof is similar to the proof of Corollary 2.16; it follows from a combination of Lemma 2.19 below (a straightforward extension of Lemma 2.14) and results from [11], as we now describe.

Assume that $A(\Gamma_0)$ is a Artin group with S as generating set. By [16, 14], the subgroup generated by any subset T of S is canonically isomorphic to the Artin-Tits group associated with the full subgraph of Γ_0. These subgroups are called the *parabolic* subgroups of $A(\Gamma_0)$. So, if $T = \{s, t\}$ with $m_{s,t} \neq \infty$ then the parabolic subgroup generated by T is a dihedral Artin group . We can define the family of (signed or unsigned) critical words of $A(\Gamma_0)$ as the (disjoint) union of the sets of (signed or unsigned) critical words of all dihedral parabolic subgroups of $A(\Gamma_0)$. The involution τ is still well defined on this set, as are the associated operations of type β on words over $S \cup S^{-1}$. So, the statement of Lemma 2.14 extends to the framework of Artin groups of large type.

Lemma 2.19. *Assume that $A(\Gamma_0)$ is an Artin group of large type. For any two words w, w', one has*
$$w \overset{\beta}{\rightsquigarrow}_R w' \implies w \overset{1,2^\star}{\rightsquigarrow}_R w'.$$

In the next section, we shall need the notion of rightward critical sequences from [11]. But we postpone the explicit definition of these to Section 3.4. For now, all we need is the following associated result, derived from [11, Proposition 3.3]

Lemma 2.20. *Assume that $A(\Gamma_0)$ is a Artin group of large type with S as generating set. Let w be a word over $S \cup S^{-1}$. If the word w is not geodesic then it can be transformed into a word that is not freely reduced by a sequence of special transformations of type β.*

With Lemma 2.19 at hand, Corollary 2.21 follows immediately.

Corollary 2.21. *Assume that $A(\Gamma_0)$ is a Artin group of large type with S as generating set. Let w be a word over $S \cup S^{-1}$. Then,*
$$w \overset{0,1,2^\star}{\rightsquigarrow}_R \varepsilon \iff \overline{w} = 1$$

Thus Property H is verified by Artin-Tits groups of large type. In fact we can prove a stronger property than H, property H^+ below, using the convexity of parabolic subgroups of Artin groups (proved for large type Artin-Tits groups in [11], but for all Artin groups in [6]):

Proposition 2.22. [6] *Assume that $A(\Gamma_0)$ is an Artin-Tits group with S as generating set. Let S_0 be included in S. If g belongs to the standard parabolic subgroup A_{S_0} then all its geodesic representative words over $S \cup S^{-1}$ are actually over $S_0 \cup S_0^{-1}$.*

Definition 2.23. Assume that (S, R) is an Artin-Tits presentation. We say that (S, R) satisfies *Property H^+* if for all $S_0 \subseteq S$ and any word w over $S \cup S^{-1}$ that represents an element of the standard parabolic subgroup A_{S_0}, there exists a word w_0 over $S_0 \cup S_0^{-1}$ so that
$$w \overset{0,1,2^\star}{\rightsquigarrow}_R w_0$$

Corollary 2.24. *Every Artin-Tits presentation of large type satisfies Property H^+.*

Proof. By Proposition 2.18 there exists a geodesic word w_0 so that $w \overset{0,1,2^\star}{\rightsquigarrow}_R w_0$. But w_0 is a geodesic word and $\overline{w_0}$ belongs to the standard A_{S_0}. So w_0 is over $S_0 \cup S_0^{-1}$. □

We note that we derive Property H^+ not simply from Property H plus convexity but rather from convexity together with the technical results from which Property H is derived. Hence it does not follow simply from convexity that Artin-Tits presentations of spherical type or of FC type satisfy Property H^+ as well as H. But nonetheless the property must hold for both (and also for sufficiently large Artin-Tits groups), since it is implied by the Property $H^\#$ that is introduced in Section 3; see the comment following Proposition 3.5.

3. Sufficiently large Artin-Tits groups

Our objective is now to prove Property H for the family of sufficiently large Artin-Tits groups, which we define in this section. Our strategy is similar to the one used in [8] to prove property H for Artin-Tits groups of FC type. Namely we prove that a stronger property $H^{\#}$ holds for large type Artin groups and then extend it to sufficiently large Artin-Tits groups using amalgamation.

3.1. **A characterization of Sufficiently large Artin-Tits groups.** Sufficiently large Artin-Tits groups were introduced in [12], with the definition expressed in terms of the Coxeter graph. Here we obtain a characterization that proves that these groups are built out of large type Artin-Tits groups and finite rank free abelian groups. We first recall the original definition of sufficiently large Artin-Tits groups.

Definition 3.1. [12] Assume that $A(\Gamma_0)$ is an Artin group with Coxeter graph Γ. We say that $A(\Gamma_0)$ is sufficiently large if the full subgraph on any triple of distinct vertices of Γ is either completely disconnected, or of large type, or contains an edge with a label ∞. In other words, if s, t, u are three distinct vertices with $m_{s,t} = 2$ and $m_{s,u} \neq 2, \infty$, then $m_{t,u} = \infty$.

From this definition we deduce that

Proposition 3.2. *The family of sufficiently large Artin-Tits groups is the smallest family of Artin-Tits groups closed under amalgamation over a standard parabolic subgroup, and which contains finitely generated free Abelian Artin-Tits groups and large type Artin groups.*

Proof. First, it is immediate from the definition that every standard parabolic subgroup of a sufficiently large Artin-Tits group is also sufficiently large.

Now, assume that A_S is a sufficiently large Artin-Tits group, and denote by S its generating set. If there is no ∞-labelled edge, then either $m_{s,t} = 2$ for all pairs s, t in S, or there is no pair s, t in S so that $m_{s,t} = 2$. So A_S is either free abelian or of large type. Assume there exist s_1, s_2 in S so that $m_{s_1,s_2} = \infty$. Then, it is well-known that A_S is canonically isomorphic to the amalgamated product $A_1 *_{A_{1,2}} A_2$ where A_1, A_2 and $A_{1,2}$ are the standard parabolic subgroups of A_S generated by $S \setminus \{s_1\}$, $S \setminus \{s_2\}$ and $S \setminus \{s_1, s_2\}$. This proves by induction on the number of ∞-labelled edges that sufficiently large Artin-Tits groups belong to the smallest family of Artin-Tits groups closed under amalgamation over a standard parabolic subgroup and which contains finitely generated free Abelian groups and Artin-Tits groups of large type. Conversely, assume that A_1, A_2 are two Artin-Tits groups , generated by S_1 and S_2 respectively ,with a common standard parabolic subgroup $A_{1,2}$ generated by $S_{1,2} = S_1 \cap S_2$. Denote by Γ_1 and Γ_2 the Coxeter graphs of A_1, A_2, respectively. Then the group $A_1 *_{A_{1,2}} A_2$ is the Artin-Tits group whose graph is obtained by glueing Γ_1 and Γ_2 along the common subgraph generated by $S_{1,2}$, and adding an ∞-labelled edge between each pair s, t of vertices so that s lies in $S_1 \setminus S_{1,2}$ and t lies in $S_2 \setminus S_{1,2}$. Therefore, if A_1, A_2 are sufficiently large Artin-Tits groups, so is $A_1 *_{A_{1,2}} A_2$. □

3.2. **The property H^+.** From the previous section, sufficiently large Artin-Tits groups are built out of large type Artin-Tits groups and free abelian groups through iterated sequences of amalgamations. So it would be natural to try and prove that sufficiently large Artin-Tits groups satisfy Property H via a proof that when two group presentations satisfy Property H and can be amalgamated in a proper way then the obtained group presentation also satisfies Property H; unfortunately, there is no clear reason why this should be true. However, since Property H is implied by Property H^+ (by considering $S' = \emptyset$), a straightforward argument proves that the amalgamated product over a standard parabolic subgroup of two Artin groups satisfying Property H^+ satisfies property H, as follows:

Proposition 3.3. *Let (S, R) be an Artin-Tits presentation. Assume $S = S_1 \cup S_2$. Set $S_{1,2} = S_1 \cap S_2$. Denote by A_1, A_2 and $A_{1,2}$ the parabolic subgroups of A_S generated by S_1, S_2 and $S_{1,2}$, respectively and by (S_1, R_1), (S_2, R_2) and $(S_{1,2}, R_{1,2})$ their Artin-Tits presentations. Assume that $A_S = A_{S_1} *_{A_{S_{1,2}}} A_{S_2}$ and (S_1, R_1), (S_2, R_2) and $(S_{1,2}, R_{1,2})$ satisfy Property H^+. Then the presentation (S, R) satisfies Property H.*

Proof. Let w be a non-empty word on $S \cup S^{-1}$. Assume $\overline{w} = 1$. We need to prove that $w \overset{0,1,2}{\rightsquigarrow}_R \varepsilon$.

We decompose w as $w = w_1 \cdots w_k$ for some minimal $k \geq 1$, such that each subword w_i is written either over $S_1 \cup S_1^{-1}$ or $S_2 \cup S_2^{-1}$ (we call the w_i the factors of w). We prove by induction on k that $w \overset{0,1,2}{\rightsquigarrow}_R \varepsilon$. Clearly the result is true when $k = 1$.

If $k \geq 2$, it follows from consideration of amalgam normal forms (see, for example [10, Corollary 1.8]) that some $\overline{w_i}$ belongs to $A_{S_{1,2}}$. Then, where w_i is written over $S_\ell \cup S_\ell^{-1}$, since (S_ℓ, R_ℓ) satisfies Property H^+, there exists w_i' in $S_{1,2} \cup S_{1,2}^{-1}$ so that $w_i \overset{0,1,2}{\rightsquigarrow}_R w_i'$. We let w' be the word derived from w by replacing w_i by w_i'. Then $w \overset{0,1,2}{\rightsquigarrow}_R w'$. If $1 < i$ then w' contains $w_{i-1}w_i'$ as a subword, and otherwise $i < k$ and w' contains $w_i'w_{i+1}$ as a subword; either subword is written over S_j. So now, since we have decomposed w' as a product of at most $k-1$ factors, induction implies that $w' \overset{0,1,2}{\rightsquigarrow}_R \varepsilon$. It follows that $w \overset{0,1,2}{\rightsquigarrow}_R w' \overset{0,1,2}{\rightsquigarrow}_R \varepsilon$, and our proof is complete. □

Unfortunately this approach does not appear to lead to a proof that all Artin-Tits groups of sufficiently large type satisfy property H, since it is not clear that the presentation of the amlgamated product would satisfy Property H^+, as we would need in order to prove that iterating the construction gives a group satisfying H. Instead we need a stronger property than H that *is* preserved under amalgamation.

3.3. The property $H^\#$.

The article [8] introduces a stronger property than Property H, namely Property $H^\#$. And in that article it is proved that the strategy proposed in Section 3.2 can be applied to Property $H^\#$. This is the way that Property H is proved to be satisfied by FC type Artin-Tits groups. We follow the same approach here to prove that Property H is satisfied by sufficiently large Artin-Tits groups.

If H is a subgroup of a group G, we call an *H-transversal* of G any subset of G that contains exactly one element from each left-H-coset, and in particular contains 1. If A_S is an Artin-Tits group generated by S, then we call an *S-sequence of transversals* any sequence $(T(S'))_{S' \subseteq S}$ for which the set $T(S')$ is a $A_{S'}$-transversal of A_S, for each $S' \subseteq S$.

Definition 3.4. Assume that (S, R) is an Artin-Tits presentation and T is an associated S-sequence of transversals. We say that (S, R) satisfies *Property $H^\#$*

(with respect to T) if, for all $S', S_0 \subseteq S$ and any word w over $S \cup S^{-1}$, setting $S'_0 = S' \cap S_0$, the following relation is satisfied:

$$\overline{w} \in A_{S'} \Rightarrow \exists v \in (S' \cup {S'}^{-1})^*,\ \exists u \in (S'_0 \cup {S'_0}^{-1})^*\ (w \overset{0,1,2}{\rightsquigarrow}_R vu \text{ and } \overline{v} \in T(S_0)).$$

Roughly speaking, this property says that for every parabolic subgroup A_{S_0} of A_S, there is an A_{S_0}-transversal $T(S_0)$ of A_S, so that one is able to find the decomposition of any element g as the product of its transversal representative and an element of A_{S_0}. Moreover, this can be done at the level of the word representatives, using operations of type $0, 1, 2$, only. Finally when the element g, itself, belongs to another parabolic subgroup $A_{S'}$ then the two obtained representative words of the decomposition can be written over $S' \cup S'^{-1}$. This imposes in particular the condition that the restriction of $T(S_0)$ to the elements that belong to $A_{S'}$ provide an $A_{S_0 \cap S'}$-transversal of $A_{S'}$. Considering the case where S' is the empty set, we get the first part of the following proposition

Proposition 3.5. [8, Proposition 2.14] *Let (S, R) be an Artin-Tits presentation.*

1. *If (S, R) satisfies Property $H^{\#}$, then (S, R) satisfies Property H.*
2. *Assume $S = S_1 \cup S_2$. Set $S_{1,2} = S_1 \cap S_2$. Denote by A_1, A_2 and $A_{1,2}$ the parabolic subgroups of A_S generated by S_1, S_2 and $S_{1,2}$, respectively and by (S_1, R_1), (S_2, R_2) and $(S_{1,2}, R_{1,2})$ their Artin-Tits presentations. Assume that $A_S = A_{S_1} *_{A_{S_{1,2}}} A_{S_2}$ and (S_1, R_1), (S_2, R_2) and $(S_{1,2}, R_{1,2})$ satisfy Property $H^{\#}$. Then (S, R) satisfies Property $H^{\#}$.*

Note also that Property $H^{\#}$ immediately implies Property H^{+}. We shall use Propositions 3.2 and 3.5 in the final section to deduce Theorems 1.4 and 1.6 from Proposition 3.13, which proves Property $H^{\#}$ for Artin-Tits groups of large type, together with the same result for finite rank free Abelian groups. Since finite rank Abelian groups are all of spherical type, and hence covered by the results of [8], we focus now on the case of large type Artin-Tits groups.

3.4. **Critical sequence.** In order to prove Property $H^{\#}$ for large type Artin-Tits groups, we have first to find a S-sequence of transversals. The key argument turns out to involve critical factorizations and associated critical sequences of transformations, which we have already mentioned, and for which we now need a definition. We recall the notion of a critical word from Section 2.4. Our definition of a critical factorization is taken from [11]. Roughly speaking a critical factorization of a word is more-or-less its expression as a concatenation of critical words with one letter overlaps.

Definition 3.6. Assume A_S is an Artin-Tits group. Let n be a positive number, and suppose that w is freely reduced.

1. We define a *critical factorization* of w of length 1 to be an expression of w as a concatenation $\alpha v_1 \beta$ of (possibly empty) words α, β and a critical word v_1, and call v_1 the *critical factor* of that factorization. A critical factorization of length 1 is considered to be both rightward and leftward.
2. For $n \geq 2$, we define a *rightward critical factorization* of w of length n with *critical factors* $v_1, \dots, v_n$ to be an expression of w as a concatenation $\alpha v_1 \cdots v_n \beta$, where v_1 is critical, and, where s is the last letter of $\tau(v_1)$ (as

defined in Section 2.4), the words $sv_2, v_3, \dots, v_n$ are the critical factors of a critical factorization of length $n-1$ of the word $sv_2v_3\cdots v_n$.

(3) For $n \geq 2$, we define a *leftward critical factorization* of w of length n with *critical factors* $v_1, \dots, v_n$ to be an expression of w as a concatenation $\alpha v_n \cdots v_1\beta$, where v_1 is critical, and, where s is the first letter of $\tau(v_1)$, the words $v_2s, v_3, \dots, v_n$ are the critical factors of a critical factorization of length $n-1$ of the word $v_nv_{n-1}\cdots v_2s$.

Note that a word w might admit many critical factorizations, both rightward and leftward, and that we do not require the subwords α, β of w to be non-empty.

It is convenient to extend our definition of τ so that it can be applied to the sequence of critical factors in a critical factorization.

(1) We define the image $\tau(v_1, \dots, v_n)$ of the sequence of critical factors of a rightward critical factorization to be the n-tuple $(v'_1, v'_2, \dots, v'_n)$. where v'_1 is the maximal proper prefix of $\tau(v_1)$, and $(v'_2, \dots, v'_n)$ is the image under τ of the sequence of critical factors $(sv_2, \dots, v_n)$ of $sv_2\cdots v_n$.

(2) We define the image $\tau(v_1, \dots, v_n)$ of the sequence of critical factors of a leftward critical factorization to be the n-tuple $(v`_1, v'_2, \dots, v'_n)$. where v'_1 is the maximal proper suffix of $\tau(v_1)$, and $(v'_2, \dots, v'_n)$ is the image under τ of the sequence of critical factors $(v_2s, \dots, v_n)$ of $v_n\cdots v_2s$.

We shall need the following later.

Lemma 3.7. *Assume that A_S is an Artin-Tits group. If $(v_1, \cdots, v_n)$ is the sequence of factors of a rightward critical factorization, then so are both $(v_1, \dots, v_k)$ and $(sv_{k+1}, \dots, v_n)$, for any $1 \leq k < n$, where s is the final letter of the last term of $\tau(v_1, \dots, v_k)$.*

Proof. This follows from the definition. We leave the details to the reader. □

Definition 3.8. Where a freely reduced word w admits a (rightward or leftward) critical factorization of length n, with critical factors $v_1, \dots, v_n$, the sequence of n critical transformations that leads to the replacement of the (rightward or leftward) concatenation of the critical factors by the (rightward or leftward) concatenation of the terms in the sequence $\tau(v_1, \dots, v_n)$ is called a *critical sequence of transformations*. That sequence is called rightward critical or leftward critical depending on whether the factorization is rightward critical or leftward critical.

We shall need the following result, derived from [11, Proposition 3.3] and [12, Proposition 3.2 (2)].

Proposition 3.9. *Let A_S be an Artin-Tits group of large type. Suppose that w is freely reduced over $S \cup S^{-1}$, representing the group element $g \in A_S$.*

(1) *If w is not geodesic then w admits a rightward critical factorization of the form $\alpha v_1 \cdots v_n\beta$, with β non-empty, and application of a single rightward critical sequence of transformations transforms w to a word $w' = \alpha v'_1 \cdots v'_n\beta$ that is not freely reduced. Specifically, the last letter of v'_n is the inverse of the first letter of β.*

(2) *Suppose that some lexicographic ordering has been chosen for $S \cup S^{-1}$, and that w_g is the minimal representative of g, with respect to the shortlex order on words*

If w is not equal to w_g, then some finite combination of rightward and leftward critical sequences of transformations together with free cancellation (after each rightward sequence) transforms w to w_g.

We shall also need the following corollary later.

Corollary 3.10. *Assume that A_S is an Artin-Tits group of Large type, and that w, w' are two geodesic representative words of the same element. Then $w' \overset{1,2}{\rightsquigarrow}_R w$.*

Proof. Set $g = \overline{w} = \overline{w'}$, fix an ordering of $S \cup S^{-1}$, and let w_g be the unique minimal representative of g with respect to the associated shortlex ordering on words. Proposition 3.9 ensures that a combination of leftward critical sequences of transformations transforms each of w, w' into w_g; hence a sequence of transformations of type β (according to Definition 2.13) transforms w and w' into w_g, and therefore w into w'. The result now follows by Lemma 2.19. □

3.5. **The S-transversal for large Type Artin-Tits groups.** With the notion of critical sequence at hand, we can now introduce an S-transversal for a large type Artin-Tits group and prove that Property $H^{\#}$ holds for Artin-Tits group of large type.

To verify that the set we define does give an S-transversal, we shall need the following.

Proposition 3.11. *Assume that A_S is an Artin-Tits group of large type. Suppose that $S_0 \subseteq S$ and $g \in A_S$. Then*

(1) *Choose $h \in A_{S_0}$ to satisfy the equation $g = g_0 h$, where g_0 is a minimal length element of gA_{S_0}. Let w_0 and z be geodesic representative words of g_0 and h, respectively. Then $w_0 z$ is a geodesic representative word of g.*
(2) *The left coset gA_{S_0} contains a unique element of minimal length.*

Proof. Let g_0, h, w_0, z be as specified in (1). It follows from the convexity (see Proposition 2.22) of parabolic subgroups of Artin groups that z is a word written over $S_0 \cup S_0^{-1}$. Since w_0 and z are freely reduced words, $w_0 z$ must also be freely reduced; for otherwise w_0 has a proper suffix over $S_0 \cup S_0^{-1}$ that freely cancels with a proper prefix of z, and hence a proper prefix of w_0 represents a shorter element of gA_{S_0} than g_0.

So now suppose that $w_0 z$ is not geodesic. Then by Proposition 3.9 $w_0 z$ admits a rightward critical factorization $\alpha v_1 \cdots v_n \beta = \alpha v \beta$ such that the last letter of the concatenation of the terms of $\tau(v_1, \ldots, v_n)$ freely cancels with the first letter s of the non-empty word β. Since w_0 and z are geodesic, clearly vs cannot be a factor of either. If v were a suffix of w_0, then we could replace that suffix by the concatenation of the terms of $\tau(v_1, \ldots, v_n)$ to get a new choice of w_0; but then we would have free reduction between w_0 and z. We deduce that v must overlap w_0 and z.

Now, if for some $k < n$ the first k terms of the critical factorization were within w_0, then by Lemma 3.7 those terms would give a critical factorization of a factor of w_0, and replacing them by the terms of $\tau(v_1, \ldots, v_k)$ would give us a different minimal choice for w_0. With this choice, Lemma 3.7 now implies that $w_0 z$ would admit a rightward critical factorization with critical factors $tv_{k+1}, \ldots, v_n$, where t is the last letter of the concatenation of the terms of $\tau(v_1, \cdots, v_k)$. Hence we can assume that the first term v_1 of the sequence overlaps both w_0 and z.

Suppose now that v_1 involves generators s_1 and t_1. Let s_1 be the rightmost letter of v_1. Then we know that s_1 is within z, so $s_1 \in S_0 \cup S_0^{-1}$. But since v_1 is not a subword of z, we see that $t_1 \notin S_0 \cup S_0^{-1}$.

Let t be the generator immediately to the right of v_1 in z (we know there must be such a generator even when $n = 1$, since β is non-empty). Then, by Lemma 2.10, t_1 is the name of the generator at the end of $\tau(v_1)$; so either $t_1 = t^{\pm 1}$ (when we have a free reduction), or t_1 is the name of a generator in v_2, which is a subword of z. Either way $t_1 \in S_0$, and so we have a contradiction.

So all cases lead to a contradiction. We conclude that $w_0 z$ must be a geodesic representative word of g.

So (1) holds. Now, (2) follows immediately. For if g_1 is another element of gA_{S_0}, then $g_1 = g_0 h_1$ for some non-trivial $h_1 \in A_{s_0}$, and so g_0 has a geodesic representative of which w_0 is a proper prefix. □

Corollary 3.12. *Assume that A_S is an Artin-Tits group of Large type. For $S_0 \subseteq S$, Denote by $T(S_0)$ the set of elements of A_S that are of minimal length in their left A_{S_0}-coset. Then, the sequence $(T(S_0))_{S_0 \subseteq S}$, is a S-sequence of transversals.*

Proof. That each $T(S_0)$ is a left transversal follows immediately from part (2) of Proposition 3.11. And $1 \in T(S_0)$ is immediate since the element 1 is minimal in its left coset A_{S_0}. □

Proposition 3.13. *Let A_S be an Artin-Tits group of Large type. Then A_S satisfies the Property $H^{\#}$.*

Proof. Let $(T(S_0))_{S_0 \subseteq S}$ be the S-sequence of transversals defined in Corollary 3.12. Choose subsets S_0, S' of S, and let w be a word with $\overline{w} \in A_{S'}$. Choose g_0 in $T(S_0)$ and h in A_{S_0} with $\overline{w} = g_0 h$, and let w_0 and z be geodesic representative words of g_0 and h, respectively.

Since $h \in A_{S_0}$, the convexity of parabolic subgroups ensures that z is written over $S_0 \cup S_0^{-1}$. By Proposition 3.11, the word $w_0 z$ is a geodesic representative word of $\overline{w}$, and so is written over $S' \cup S'^{-1}$. Hence, w_0 is written over $S' \cup S'^{-1}$, and z is written over $S'_0 \cup S_0'^{-1}$, where $S'_0 = S' \cap S_0$. So, in order to prove that Property $H^{\#}$ holds for A_S, it remains to prove that $w \overset{0,1,2}{\rightsquigarrow}_R w_0 z$.

By proposition 2.18, there exists a geodesic representative word w' so that $w \overset{0,1,2}{\rightsquigarrow}_R w'$. It remains to show that $w' \overset{1,2}{\rightsquigarrow}_R w_0 z$. Since w' and $w_0 z$ are two geodesic representative words of the same element, this follows immediately from Corollary 3.10. □

3.6. **Proof of Theorems 1.6 and 1.4.** We are now almost ready to prove Theorem 1.6. Using Proposition 3.5, we deduce the result from Proposition 3.13 by induction on the number of edges labelled with ∞ in the Coxeter graph. We need a preliminary result

Lemma 3.14. *Let (S_1, R_1) and (S_2, R_2) be two Artin-Tits presentations that satisfy property $H^{\#}$ with $S_1 \cap S_2 = \emptyset$. Set*

$$R_{1,2} = \{s_1 s_2 = s_2 s_1 \mid s_1 \in S_1;\ s_2 \in S_2\}$$

Then, the Artin-Tits presentation $(S_1 \cup S_2, R_1 \cup R_2 \cup R_{1,2})$ satisfies property $H^{\#}$.

Note that in the above lemma, the Artin-Tits group $A_{S_1 \cup S_2}$ associated with the presentation $(S_1 \cup S_2, R_1 \cup R_2 \cup R_{1,2})$ is isomorphic to the direct product $A_{S_1} \times A_{S_2}$.

Proof. Let $(T(S_{1,0}))_{S_{1,0}\subseteq S_1}$, be a S_1-sequence of transversals and $(U(S_{2,0}))_{S_{2,0}\subseteq S_2}$, be a S_2-sequence of transversals. For $S_0 \subseteq S_1 \cup S_2$, set

$$V(S_0) = \{g_1 g_2 \mid g_1 \in T(S_0 \cap S_1); g_2 \in U(S_0 \cap S_2)\}.$$

Since $A_{S_1\cup S_2}$ is the direct product of A_{S_1} and A_{S_2}, it immediately follows that $(V(S_0))_{S_0\subseteq S}$, is a S-sequence of transversals.

Let S' be a subset of S. Set $S'_1 = S_1 \cap S'$, $S'_2 = S_2 \cap S'$, $S'_{1,0} = S_1 \cap S' \cap S_0$ and $S'_{2,0} = S_2 \cap S' \cap S_0$. Let w be a word in $S \cup S^{-1}$ so that $\overline{w}$ belongs to $A_{S'}$.

Since the latter group is the direct product of $A_{S'_1}$ and $A_{S'_2}$, there exists a unique pair $(g_1 g_2)$ so that $\overline{w} = g_1 g_2$ with $g_1 \in A_{S'_1}$ and $g_2 \in A_{S'_2}$. Note that for any two words w', w'', any s_1 in S_1 and s_2 in S_2, and $\varepsilon, \tau \in \{\pm 1\}$, any transformation from $w' s_i^{\varepsilon} s_j^{\tau} w$" to $w' s_j^{\tau} s_i^{\varepsilon} w$" is an elementary transformation of type 1 or of type 2, since $s_1 s_2 = s_2 s_1$ belong to $R_{1,2}$ Therefore, there exist w_1 in $S_1 \cup S_1^{-1}$ and w_2 in $S_2 \cup S_2^{-1}$ so that $w \overset{1,2}{\rightsquigarrow}_R w_1 w_2$. This imposes $\overline{w_1} = g_1$ and $\overline{w_2} = g_2$.

Since (S_1, R_1) satisfies property $H^{\#}$, there exist $v_1 \in (S'_1 \cup {S'_1}^{-1})^*$ and $u_1 \in (S'_{1,0} \cup {S'_{1,0}}^{-1})^*$ so that $w_1 \overset{0,1,2}{\rightsquigarrow}_R v_1 u_1$ and $\overline{v_1} \in T(S_{1,0})$. Similarly, there exist $v_2 \in (S'_2 \cup {S'_2}^{-1})^*$ and $u_2 \in (S'_{2,0} \cup {S'_{2,0}}^{-1})^*$ so that $w_2 \overset{0,1,2}{\rightsquigarrow}_R v_2 u_2$ and $\overline{v_2} \in T(S_{2,0})$. Thus, we get $w \overset{0,1,2}{\rightsquigarrow}_R v_1 u_1 v_2 u_2$. By the same argument as before we have $u_1 v_2 \overset{1,2}{\rightsquigarrow}_R v_2 u_1$ and $w \overset{0,1,2}{\rightsquigarrow}_R v_1 v_2 u_1 u_2$. The word $v_1 v_2$ lies in $(S' \cup {S'}^{-1})^*$ and by definition of $V(S_0)$, the element $\overline{v_1 v_2}$ belongs to this transversal. Finally, the word $u_1 u_2$ lies in $(S'_0 \cup {S'_0}^{-1})^*$. Hence, Property $H^{\#}$ holds for the presentation $(S_1 \cup S_2, R_1 \cup R_2 \cup R_{1,2})$. □

Proof of Theorem 1.6. Recall that $\mathcal{T}$ is the family of those Artin-Tits presentation (S, R) whose Coxeter graph Γ satisfies the property that every connected full subgraph without an ∞-labelled edge is either of spherical type or of large type. For (S, R) in $\mathcal{T}$, we denote the number of edges labelled with ∞ in the associated graph by $d_\infty(S, R)$. Let (S, R) belong to $\mathcal{T}$ and denote by Γ its Coxeter graph. As announced in the introduction of this section, we prove the result by induction on $d_\infty(S, R)$.

If $d_\infty(S, R) = 0$, then T is a union of connected Coxeter graphs of spherical type or of large type. Since Property $H^{\#}$ holds for both types (see [8] for the spherical case), by Lemma 3.14, (S, R) satisfies Property $H^{\#}$.

Now assume $d_\infty(S, R) \geq 1$. Let s_1, s_2 belong to S so that $m_{s_1,s_2} = \infty$. Set $S_1 = S \setminus \{s_2\}$, $S_2 = S \setminus \{s_1\}$ and $S_{1,2} = S \setminus \{s_1, s_2\}$. Consider (S_1, R_1), (S_2, R_2) and $(S_{1,2}, R_{1,2})$, the Artin-Tits presentations associated with the full subgraphs of Γ generated by S_1, S_2 and $S_{1,2}$, respectively. They are the presentations of the subgroups A_1, A_2 and $A_{1,2}$ of A_S generated by S_1, S_2 and $S_{1,2}$, respectively, and A_S is equal to the amalgamated product $A_1 *_{1,2} A_2$. Since (S, R) belongs to $\mathcal{T}$, the presentations (S_1, R_1), (S_2, R_2) and $(S_{1,2}, R_{1,2})$ belong to $\mathcal{T}$, too. Moreover, we have $d_\infty(S_1, R_1) < d_\infty(S, R)$, $d_\infty(S_2, R_2) < d_\infty(S, R)$ and $d_\infty(S_{1,2}, R_{1,2}) < d_\infty(S, R)$. So by the induction hypothesis, Property $H^{\#}$ hold for (S_1, R_1), (S_2, R_2) and $(S_{1,2}, R_{1,2})$. So, (S, R) verifies Property $H^{\#}$ by Proposition 3.5. It also satisfies Property H by 3.5. □

Theorem 1.4 can be deduced as an immediate corollary, using Proposition 3.2 and the fact that every Artin-Tits presentation of large type belongs to the family $\mathcal{T}$.

References

[1] **K.I. Appel.** *On Artin groups and Coxeter groups of large type.* Contemp. Math. **33** (1984), 50–78.

[2] **K.I. Appel, P.E. Schupp** *Artin groups and infinite Coxeter groups.* Invent. Math. **72** (1983), 201–220.

[3] **T. Brady, J.P. McCammond.** *Three-generator Artin groups of large type are biautomatic.* P. Pure Appl. Algebra **151** (2000), 1–9.

[4] **E. Brieskorn, K. Saito.** *Artin-Gruppen und Coxeter-Gruppen.* Invent. Math. **17** (1972), 245–271.

[5] **R. Charney** *Injectivity of the positive monoid for some infinite type Artin groups.* Geometric group theory down under (Canberra, 1996), 103–118, de Gruyter, Berlin, 1999.

[6] **R. Charney, L Paris.** *Convexity of parabolic subgroups in Artin groups* Bull. Lond. Math. Soc. **46** (2014), 1248–1255.

[7] **J. Crisp, E. Godelle, B. Wiest** *The conjugacy problem in right-angled Artin groups and their subgroups* J.P.A.A. **2** (2009), 442–460.

[8] **P. Dehornoy, E. Godelle** *A conjecture about Artin-Tits groups* J.P.A.A. **217** (2013), 241–256.

[9] **P. Dehornoy** *The subword reversing method* Intern. J. Alg. and Comput. **21** (2011), 71–118.

[10] **E. Godelle** *Parabolic subgroups of Artin groups of type FC* Pacific J. Math. **208** (2003), 243–254.

[11] **D.F. Holt, S. Rees** *Artin groups of large type are shortlex automatic with regular geodesics* Proc. London Math. Soc. **104** (2012), 486–512.

[12] **D.F. Holt, S. Rees** *Shortlex automaticity and geodesic regularity in Artin groups*, Groups, Complex. Cryptol. 5 (2013), 1–23.

[13] **J. Mairesse, F. Mathéus.** *Growth series for Artin groups of dihedral type.* Int. J. Algebra Comput. **16** (2006), 1087–1107.

[14] **L. Paris.** *Parabolic subgroups of Artin groups.* J. Algebra. **196** (1997), 369–399.

[15] **H Servatius**, *Automorphisms of Graph Groups*, J. Algebra **126** (1987), 34-60.

[16] **H. van der Lek.** *The homotopy type of complex hyperplane complements.* Ph. D. Thesis, Nijmegen, 1983.

Eddy Godelle, Normandie Univ, France; UNICAEN, LMNO, F-14032 Caen, France; CNRS UMR 6139, F-14032 Caen, France

E-mail address: `eddy.godelle@unicaen.fr`

URL: `//www.math.unicaen.fr/∼godelle`

Sarah Rees, School of Mathematics and Statistics, University of Newcastle, Newcastle NE1 7RU, UK

E-mail address: `sarah.rees@ncl.ac.uk`

URL: `//www.mas.ncl.ac.uk/∼nser`