Introduction

Proof?

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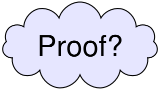
INTRODUCTION



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Can you explain why these are true?

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$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota\text{‘}x . \beta = \iota\text{‘}y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

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From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

The proof that $1 + 1 = 2$ is completed in Volume II, page 86, with the comment, *“The above proposition is occasionally useful.”*

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Experiments suggest that answer is <u>always</u> 1089.

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To be sure, we need a proof.

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$$\begin{aligned}(100x+10y+z)+(100z+10y+x) &= 900+(90+90)+9\\ &= 1089.\end{aligned}$$

Why proof?

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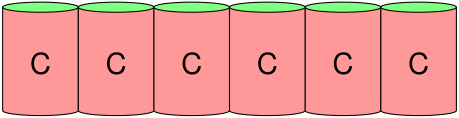

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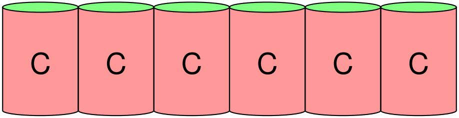

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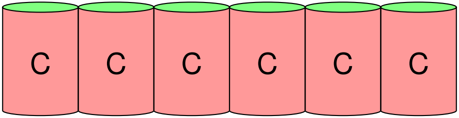

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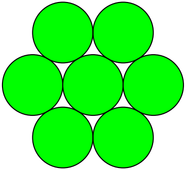

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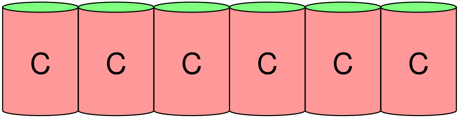

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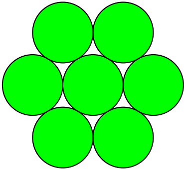

Exercise. Check this.

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It is enough to find area on top. Assume cans have radius=1.

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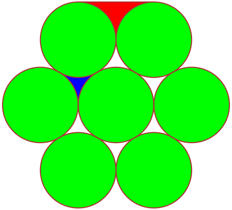
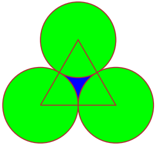
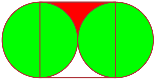




Blue area = $\sqrt{3} - \frac{\pi}{2}$



Red area = $\frac{1}{2}(2 \times 2 - \pi)$

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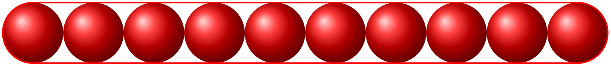
Answer. Arrange them in a line, to get a sausage shaped wrapping.



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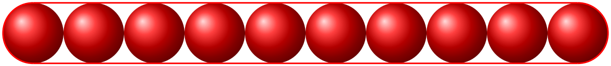

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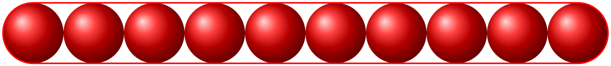

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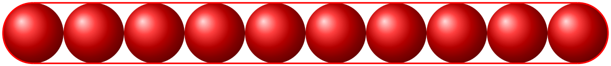

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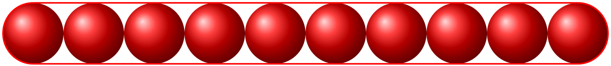

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These examples illustrate the need for careful proof to be sure that a statement is always true.

One more example ...

$$\pi$$

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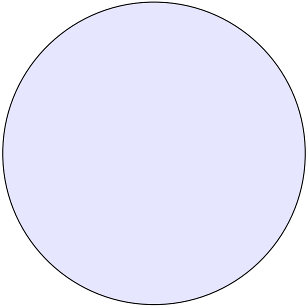
Circle of diameter 1

Circumference = π



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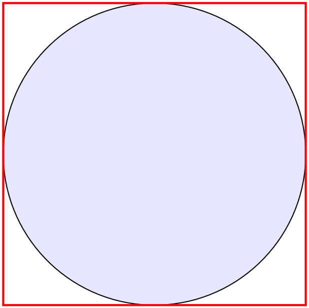

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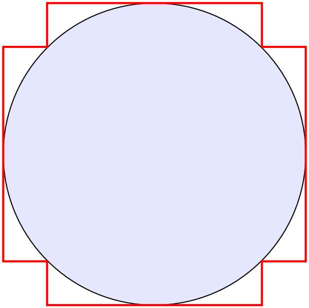

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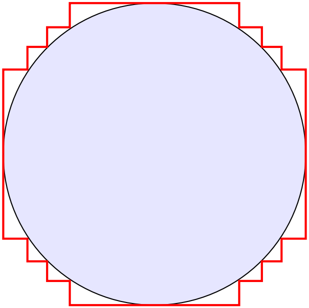

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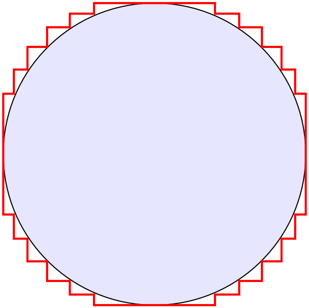

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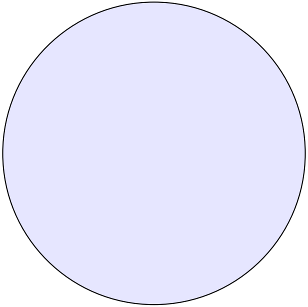

Continue in this way to get $\pi = 4$?

COMMON EQUIVALENCES

Equivalence	Name
$p \longleftrightarrow q \Longleftrightarrow (p \longrightarrow q) \wedge (q \longrightarrow q)$	Equivalence law
$p \longrightarrow q \Longleftrightarrow \neg p \vee q$	Implication law
$\neg\neg p \Longleftrightarrow p$	Double Negation law
$p \wedge p \Longleftrightarrow p$ and $p \vee p \Longleftrightarrow p$	Idempotent laws
$p \wedge q \Longleftrightarrow q \wedge p$ and $p \vee q \Longleftrightarrow q \vee p$	Commutative laws
$(p \wedge q) \wedge r \Longleftrightarrow p \wedge (q \wedge r)$	Associative law I
$(p \vee q) \vee r \Longleftrightarrow p \vee (q \vee r)$	Associative law II
$(p \wedge q) \vee r \Longleftrightarrow (p \vee r) \wedge (q \vee r)$	Distributative law I
$(p \vee q) \wedge r \Longleftrightarrow (p \wedge r) \vee (q \wedge r)$	Distributativ law II
$\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$	De Morgan's law I
$\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$	De Morgan's law II
$p \wedge (p \vee q) \Longleftrightarrow p$	Absorbtion law I
$p \vee (p \wedge q) \Longleftrightarrow p$	Absorbtion law I
$p \longrightarrow q \Longleftrightarrow \neg p \longrightarrow \neg q$	Contrapositive law

Common rules of Inference

Rule of Inference	Name
$\left.\begin{array}{l} p \\ p \longrightarrow q \end{array}\right\} \Longrightarrow q$	Modus Ponens
$\left.\begin{array}{l} \neg q \\ p \longrightarrow q \end{array}\right\} \Longrightarrow \neg p$	Modus Tollens
$\left.\begin{array}{l} p \longrightarrow q \\ q \longrightarrow r \end{array}\right\} \Longrightarrow p \longrightarrow r$	Transitivity
$p \wedge q \Longrightarrow q$	Simplification
$p \Longrightarrow p \vee q$	Addition

4.Induction

Proof by induction is a method of proving that a sequence of statements are all true.

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Induction can be used to prove for example that, for all integers $n \geq 1$,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2,$$

or that, for all integers $n \geq 1$,

$$n! \leq n^n.$$

In the first case above the task is to prove that the entries in the second and third columns of the table below are equal.

n	sum of first n odd numbers	n^2
1	1	1×1
2	$1+3=4$	2×2
3	$1+3+5=9$	3×3
4	$1+3+5+7=16$	4×4
5	$1+3+5+7+9=25$	5×5
$\vdots$	$\vdots$	$\vdots$
100	$1+\cdots+199$	10000
$\vdots$	$\vdots$	$\vdots$
k	$1+\cdots+(2k-1)$	k^2
$\vdots$	$\vdots$	$\vdots$

THE GENERAL CASE

In both cases, a predicate $P(n)$ is given, and induction is used to prove

$$\forall n \in \mathbb{N}\ P(n),$$

or equivalently

$$\{n \in \mathbb{N} \mid P(n)\} = \mathbb{N}$$

In the first case

$$P(n) \text{ is } 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

and in the second case $P(n)$ is

$$P(n) \text{ is } n! \leq n^n.$$

THE IDEA

Recall that the set of natural numbers $\mathbb{N}$ is the set $\{1, 2, \cdots\}$ of positive integers.

Induction is based on a fundamental property of $\mathbb{N}$:

THEOREM 4.1

A subset S of $\mathbb{N}$ which satisfies both

1. $1 \in S$ *and*
2. *for all $n \in \mathbb{N}$, if $n \in S$ then $n+1 \in S$;*

is equal to $\mathbb{N}$.

[This follows from an axiom for $\mathbb{N}$, called “Well-Ordering”. More details will be covered in MAS1702.]

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[This follows from an axiom for $\mathbb{N}$, called “Well-Ordering”. More details will be covered in MAS1702.]

Using Theorem 4.1, to check that a given set S is equal to $\mathbb{N}$, it is enough to verify that both 4.1.1 and 4.1.2 above hold; which is what is done in a proof by induction.

The Principle of Induction

In the Theorem below $P(n)$ is a predicate defined for all integers $n \geq 1$.

For example $P(n)$ might be, "the sum of the first n odd positive integers equals n^2", as in the first example above,

or it could be "$n! \leq n^n$", as in the second example.

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Theorem 4.2

Let $P(n)$ be a predicate defined for all integers $n \geq 1$. Suppose that

(1) *$P(1)$ is true, and*

(2) *For arbitrary $k \in \mathbb{N}$,*
$P(k) \implies P(k+1)$.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

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From Theorem 4.1, it follows that $S = \mathbb{N}$, so $P(n)$ is true for all $n \in \mathbb{N}$.

Example 4.3

Prove by induction that $1+2+\cdots+n = \frac{n(n+1)}{2}$, for all integers $n \geq 1$.

In this case $P(n)$ is $1+2+\cdots+n = \frac{n(n+1)}{2}$.

The first step is to show that $P(1)$ is true.
$P(1)$ is $1 = (1\times 2)/2$, so is true.

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The next step is to show that, for an arbitrary k, if $P(k)$ is true then $P(k+1)$ is true.

Example 4.3

Prove by induction that $1+2+\cdots+n=\frac{n(n+1)}{2}$, for all integers $n \geq 1$.

In this case $P(n)$ is $1+2+\cdots+n=\frac{n(n+1)}{2}$.

The first step is to show that $P(1)$ is true.
$P(1)$ is $1=(1\times 2)/2$, so is true.

The next step is to show that, for an arbitrary k, if $P(k)$ is true then $P(k+1)$ is true.

$P(k)$ is $1+2+\cdots+k=\frac{k(k+1)}{2}$:
it's obtained by replacing n by k throughout $P(n)$.

We assume $1+2+\cdots+k=\frac{k(k+1)}{2}$ and must show that $P(k+1)$ is true: that is

$$1+2+\cdots+(k+1)=\frac{(k+1)((k+1)+1)}{2}.$$

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$$= \frac{k^2+k+2(k+1)}{2}$$

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\begin{aligned}
&= \frac{k(k+1)}{2} + (k+1), \text{ (as } P(k) \text{ is true} \\
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&= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}.
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Therefore $P(k+1)$ holds.

Conclusion. By induction, $P(n)$ holds for all $n \geq 1$.

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Prove by induction that

$$1+3+5+\cdots+(2n-1)=n^2$$

for all $n \geq 1$.

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Since this is true the first part of the proof is complete.

The inductive hypothesis (IH)

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The inductive step

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EXAMPLE 4.5

Prove by induction that $n! \leq n^n$, for all integers $n \geq 1$.

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The IH: Assume that $P(k)$ holds, for some k.

That is $k! \leq k^k$.

Inductive step:
Show that $P(k+1)$ holds. That is, show that
$(k+1)! \leq (k+1)^{(k+1)}$.

Solution, cont.

$$\begin{aligned}(k+1)! &= (k!)(k+1)\\ &\leq k^k(k+1) \text{ using } k! \leq k^k\\ &\leq (k+1)^k(k+1)*\\ &= (k+1)^{(k+1)}.\end{aligned}$$

Thus $(k+1)! \leq (k+1)^{(k+1)}$.

We have shown that $P(k) \Rightarrow P(k+1)$.

Conclusion: By induction, $n! \leq n^n$, for all $n \geq 1$. $\square$

Solution, cont.

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$$\leq (k+1)^k(k+1)*$$
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We have shown that $P(k) \Rightarrow P(k+1)$.

Conclusion: By induction, $n! \le n^n$, for all $n \ge 1$. □

Recap. PROOF BY INDUCTION

Theorem 4.2. Let $P(n)$ be a predicate, defined for all $n \in \mathbb{N}$. Suppose that

$P(1)$ is true, and

$P(k) \implies P(k+1)$, for arbitrary $k \in \mathbb{N}$.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

REMARKS

- In proof by induction we make the assumption that $P(k)$ holds for an arbitrary $k \geq 1$ and then prove that $P(k+1)$ also holds. For the proof to be correct we must be sure this works for all possible values of k (which is what is meant by “arbitrary”). If it fails for just one value of k then the proof does not work.

Remarks

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- Induction is a powerful method of proof, but sometimes does not give insight into why a result is true.

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- Induction is a powerful method of proof, but sometimes does not give insight into why a result is true.
  Can we understand better why Example 4.4 is true?

Example 4.4 says: $1+3+5+\cdots+(2n-1)=n^2$, for all $n \geq 1$.

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Let $s = 1+3+\cdots+(2n-3)+(2n-1)$ (n terms).

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This proof gives more insight.

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This proof gives more insight.

On the other hand, the proof by induction in Example 4.5 does shed light on why the result holds.

Summation notation

Note: to save space, write

$$1+3+5+\cdots+(2n-1)=\sum_{j=1}^{n}(2j-1)$$

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This notation is used in exercises.

Example. For all $n \geq 1$, $\sum_{j=1}^{n}(2j-1) = n^2$. [Repeated!]

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Therefore $P(n)$ holds, for all $n \geq 1$.

Example 4.6

Prove by induction that, for all $n \in \mathbb{N}$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

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$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

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$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

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This is the right side of $P(k+1)$. So $P(k+1)$ is true.

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which simplifies to $\dfrac{(k+1)(k+2)(2k+3)}{6}$.

This is the right side of $P(k+1)$. So $P(k+1)$ is true.

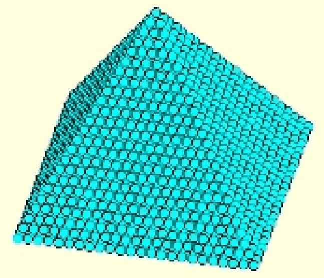
So $P(n)$ is true for all $n \in \mathbb{N}$.

24

$$1^2 + 2^2 + 3^2 + \cdots + 24^2 = 4900 = 70^2$$

In 1875, the French mathematician Édouard Lucas challenged his readers to prove this:

A square pyramid of cannon balls contains a square number of cannon balls only when it has 24 cannon balls along its base.



In other words, the only solution of

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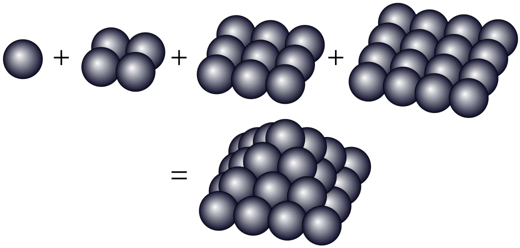
where m, n are integers greater than 1 is $n = 24$.



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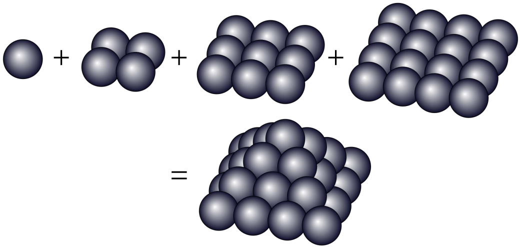

The first proof was given in 1918 by G. N. Watson.

This looks like a curiosity, but the solution leads to a very dense packing of spheres in 24 dimensions. It is also used in physics: bosonic string theory in 26 dimensions.
Key words: Leech lattice, Monster group.

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$$\sqrt{1^2+2^2+3^2+\cdots+24^2}=70.$$

This only happens in 24-dimensions.

Example 4.7

An infinite sequence $x_1, x_2, x_3, \ldots$ of integers is defined by the rules $x_1 = 2$ and $x_{n+1} = x_n + 2(n+1)$, for all $n \geq 1$. Show by induction that $x_n = n(n+1)$, for all $n \in \mathbb{N}$.

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Therefore $P(n)$ holds for all $n \in \mathbb{N}$.

Theorem 3.2. Let $P(n)$ be a predicate, defined for all $n \in \mathbb{N}$. Suppose that

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Conclusion: $P(n)$ holds for all $n \geq 10$.

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The Fibonacci numbers are the elements of the sequence $f_1, f_2, f_3, \ldots$ generated by the rules

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FIBONACCI NUMBERS

The Fibonacci numbers are the elements of the sequence $f_1, f_2, f_3, \ldots$ generated by the rules

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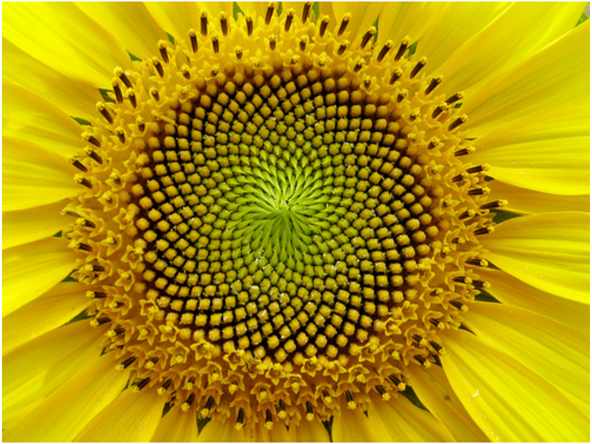
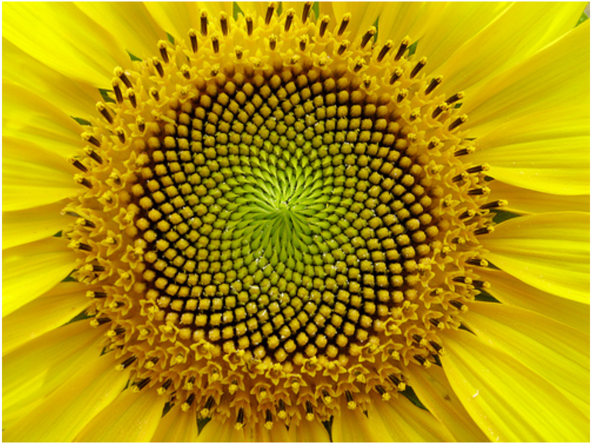
The sequence is named after the Italian Fibonacci, who introduced the sequence in 1202 AD, although the it had been described earlier by Indian musicians (Virahanka, 700 AD). The sequence appears in many places in mathematics as well as in biology: DNA, trees, leaves, cones.

21



21



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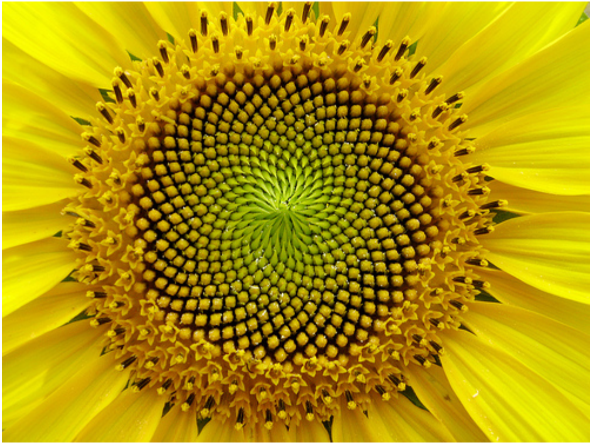
21



21 anticlockwise spirals. 34 clockwise.

More Fibonacci numbers in nature

• primrose, buttercup	**8**
• corn marigold, cineria	**13**
• black eyed Susan, chicory	**21**
• daisies	**13, 21, 34**
• pine cone spirals	**8, 13**
• sunflower spirals	**21,34,55,...233.**

EXAMPLE 4.9

If we take every third Fibonacci number we obtain a new sequence of numbers,

$$f_3, f_6, f_9, f_{12}, \ldots$$

with values

$$2, 8, 34, 144, 610, 2584, 10946, 46368, 196418, \ldots$$

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Prove, by induction that f_{3n} is even, for all $n \geq 1$.

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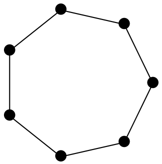
By induction, $P(n)$ holds for all $n \geq 1$.

A geometric example of induction

Example 4.10

Choose n points on a circle and connect them in order to produce a polygon. Show, by induction, that the interior angles add to $180(n-2)$ degrees, for $n \geq 3$.



A GEOMETRIC EXAMPLE OF INDUCTION

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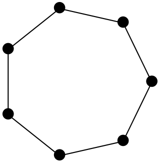

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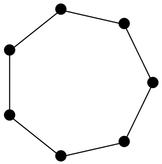

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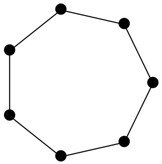

Let $P(n)$ be the statement about interior angles.
$P(3)$ is true: the angles of a triangle add to 180 degrees.
Assume that $P(k)$ is true: the interior angles a polygon with k vertices add to $180(k-2)$ degrees.

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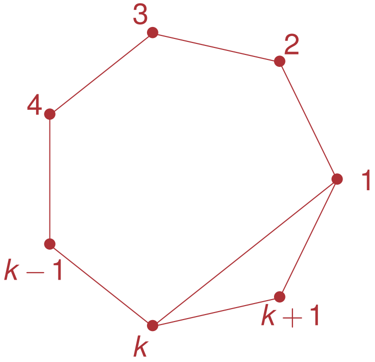

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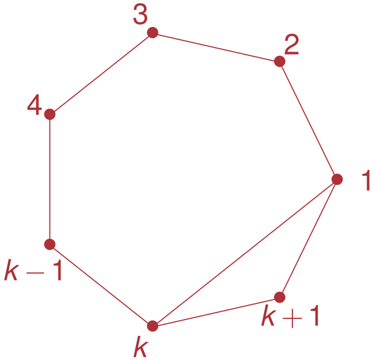

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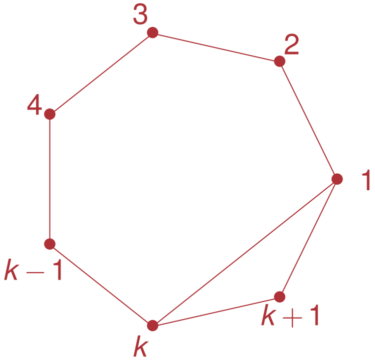

This increases the sum of the interior angles by 180 degrees, giving $180(k-1)$ degrees. Therefore $P(k+1)$ is true.

So $P(n)$ is true for all $n \geq 3$.

Objectives

You should now be able to:

(i) understand the principle of proof by induction;

(ii) carry out proof by induction, both starting with the integer 1 and starting with an integer other than 1.