UNIVERSITY OF NEWCASTLE UPON TYNE

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 Mock

MAS345

Algebraic Geometry of Curves

Time: 1 hour 30 minutes

Credit will be given for all answers to questions in Section A, and for the best TWO answers to questions in Section B.

No credit will be given for other answers and students are strongly advised not to spend time producing answers for which they will receive no credit.

Marks allocated to each question are indicated.

There are FOUR questions in Section A and THREE questions in Section B.

Section A

A1. Let l_1 be the affine line with equation $2x - 7y - 3 = 0$ and l_2 be the affine line with equation $5x + y - 6 = 0$.

(a) Find the equation and parametric form of the line l parallel to l_2 through the point $(2, 1)$.

(b) Find the point of intersection of l and l_1.

9 marks

A2. Let f be the polynomial $f = 3y^2 + x^3 - y - 2x^2$ and let C be the affine curve with equation $f = 0$.

(a) Find the intersections of C with the following lines and the corresponding intersection numbers.

i. The line l_0 with equation $x + y = 0$.

ii. The line l_1 with equation $x = 0$.

iii. The line l_2 with equation $y = 0$.

(b) Write down the homogenization F of the polynomial f and let D be the projective curve with equation $F = 0$. Let L be the projective line with equation $x = 0$. Find the points of intersection of D with L and the corresponding intersection numbers. Say which points of $D \cap L$ correspond to points of $C \cap l_1$ (and to which ones) and which do not.

12 marks

A3. Show that the polynomial $f = x^3 + y^3 + xy$ is irreducible.

10 marks

A4. Let C be a curve of degree d and suppose that C has m singularities lying on a line l.

(a) Show that if $2m > d$ then $l \subseteq C$. (If you use any results from the course, say so.)

(b) Show that if C is irreducible then $2m \leq d$.

9 marks

Section B

B5. Let

$$f(x, y) = (x^2 + y^2)^2 - x^2y - y^3$$

and let C be the complex curve with equation $f = 0$.

(a) Find all the singular points of C.

(b) Find the multiplicity of each singular point of C.

(c) Find parametric forms (or equations if you prefer) for all the tangents to C at all singular points.

(d) Write down the homogenization F of f. Find all the singular points of the curve with equation $F = 0$ and their multiplicities.

30 marks

B6. Let C and D be projective curves of degree n which intersect in exactly n^2 points. Assume that precisely mn of these points lie on an irreducible curve E of degree m, with $m < n$. Let f, g and h be the polynomials defining C, D and E respectively.

(a) Let $(a : b : c)$ be a point of E which is not in $C \cap D$. Let $\lambda = g(a, b, c)$ and $\mu = -f(a, b, c)$. Show that the polynomial $s = \lambda f + \mu g$ has degree n and that $C_s \cap E$ contains at least $nm + 1$ points (where C_s is the curve with equation $s = 0$).

(b) Show, quoting any major theorems that you use, that E and C_s have a common component. Conclude that E is a component of C_s.

(c) Show that $s = ht$ for some polynomial t of degree at most $n - m$.

(d) Prove that the remaining $n(n - m)$ points of $C \cap D$ lie on a curve of degree at most $n - m$.

30 marks

B7. (a) Define a *point of inflexion* of a projective curve.

(b) Define the *Hessian* of the curve C with equation $f = 0$.

(c) Let

$$f = x^3 + y^3 + z^3$$

and let C be the complex curve with equation $f = 0$.

i. Show that C is non–singular and find all its points of inflexion.

ii. The Group Law is defined on C, taking the identity element O to be the inflexion $(0 : -1 : 1)$. Let $A = (-1 : 0 : 1)$ and $B = (-\omega : 0 : 1)$, where $\omega = e^{i2\pi/3}$. Find the point $A + B$ of C.

iii. Given that the Group Law is defined on C as in part (cii) above, prove (using properties of curves) that the inverse of the identity O is O.

30 marks