MAS2216/MAS3216 Enumeration and Combinatorics

Assignment 1

Maple Worksheet 1, Week 3: Rook Polynomials

MAS2216 Last Names A–N: Wednesday 14th October, Daysh, Brae PC
MAS2216 Last Names O–Z: Friday 16th October, Herschel, Loch.Pond.Bank
MAS3216: Friday 16th October, Herschel Building Loch.Pond.Bank PC

Complete Maple TA "Rook Polynomial" assignment before leaving the practical. This worksheet forms part of Assignment 1 and you can only do it during this hour.

1 Introduction

Rook Polynomials can be used to find the number of solutions to various scheduling problems that involve permutations, subject to given restrictions. The problems that can be addressed this way can all be described in terms of rooks on a chessboard (see for example Questions 5 and 6 on the Rook Polynomial Maple TA assignment). In this handout rook polynomials are described and their use in such problems illustrated.

Working through the problems on the handout, using the Answers and Hints given, will enable you to answer the Maple TA assignment questions, which form part of Assignment 1. Do not hand in answers to the questions on this handout but enter Maple TA and go into the class "MAS2216 Assignment Worksheets" (or "Enumeration and Combinatorics-Stage 3" if you do MAS3216) and select the Rook Polynomial assignment to answer the assignment questions.

Section 2 covers questions similar to the first 4 assessed questions in the Rook Polynomial Maple TA assignment and the two following Sections give solutions to these questions as well as hints for Questions 5 and 6.

2 Problems (not to be handed in)

In chess a rook can move horizontally or vertically in any direction any distance on the board. Two rooks are said to be "attacking" each other if they are both in the same row or column; so one can get, in a single move, to the position occupied by the other. In the problems below assume that all rooks look alike.

1. Suppose that chess is played on a 12 x 12 board. In how many different ways can 8 rooks be placed on the board so that none of the rooks are attacking each other?

2. Suppose that you have an 12 x 12 board. One of the corner squares is removed. No rook can be placed there. Given this, in how many ways can 12 rooks be placed on the board so that no rooks are attacking each other?

3. Suppose that you have a 12 x 12 board. In how many ways can you place 11 rooks so that no rooks are attacking each other?

4. Suppose you have the following board where rooks can only be placed on white squares.
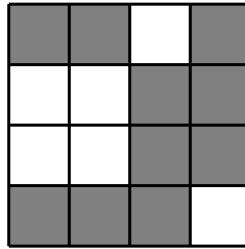
   (a) How many ways are there to place no rooks on the board?

   (b) How many ways are there to place one rook on the board?

   (c) How many ways are there to place 2 rooks on the board so that none are attacking each other?

   (d) How many ways are there to place 3 rooks on the board such that none are attacking each other?

   (e) How many ways are there to place 4 rooks on the board such that none are attacking each other?

   (f) How many ways are there to place 5 or more rooks on the board such that none are attacking each other?

3   Answers and Hints

1. Each row and each column can contain at most one rook. To fit 12 of them onto a 12×12 board I must therefore put one in each row and column. For the rook in the first column, assuming I position this one first, there are 12 rows to choose from. If I put the next rook in the second column there are only 11 rows to choose from and so on. This gives 12! ways to place the rooks on the board. That is 479001600.

2. Place a rook on the row with a corner missing. There are 11 ways to do this. After that I can think of the rest of the board as an 11×11 board. I need to place 11 rooks on this board and there are $11! = 39916800$ ways using the same reasoning as in Question 1. This gives a total of $11 \times 11! = 43084800$ ways to place the rooks. (Alternatively: there are 11! ways to position the 12 rooks if one of them **is** on the missing square. Since there are 12! ways to place rooks when there is no missing square there are $12! - 11! = (12 - 1) \times 11! = 11 \times 11!$ ways to place rooks without using the missing square.)

3. The task can be split into two parts. Since we have 11 rooks to place we must use 11 rows and 11 columns. First we shall choose which rows to use, then we shall choose where to position the rook in each row we have chosen.

   The number of ways of choosing 11 out of 12 rows is $\binom{12}{11} = 12$. (Given any two different choices of 11 rows we will end up with different positionings for the rooks.) Given a choice of 11 rows there are 12 positions (columns) in which we can put the rook in the first row. Then there are 11 choices for the rook in the second row, and so on. The total number of ways of position the 11 rooks, once the rows have been chosen is therefore $12 \times \cdots \times 2 = 12!$. The total number of ways of positioning the rooks is therefore

$$12 \times 12! = 5748019200$$

   (The same argument gives $\binom{12}{k} \times \frac{12!}{(12-k)!}$ ways to position $k \leq 12$ non-attacking rooks on the board.)

4. Log in to Maple, open a worksheet and load the `combinat` package by typing

   `>with(combinat);`

   Now read in the required Maple code by typing

   `>read "Q:/2216/rookpolynomial.m";`

   (If you have not mapped the `Q` drive this will not work. In this case see the instructions at `http://www.ncl.ac.uk/math/internal/coursedata.htm`.)

   Now define the set of shaded squares in maple using the following command, where `[1,1]` represents the upper left corner and `[1,4]` the lower left corner etc.

   `B:=[[1,1],[1,2],[1,4],[2,3],[2,4],[3,3],[3,4],[4,1],[4,2],[4,3]]:`

   Now we use the rookpolynomial function.

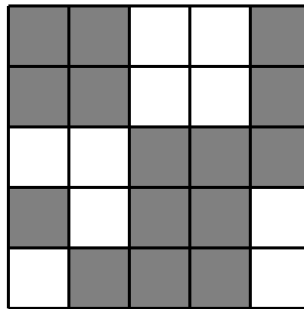   `rookpolynomial(B,4,x);`

   This will give us the rook polynomial

$$1 + 6x + 11x^2 + 8x^3 + 2x^4$$

   for this board. B is the shaded area of the board. The number 4 represents the size of the board: it is a 4×4 board. The output is a polynomial of degree four. The coefficient of x^d gives the number of ways d rooks can be placed on the board so that no two are attacking each other. So there are 2 ways of placing 4 rooks on the board; 8 ways of placing 3 rooks on the board and so on. Of course there is only one way of placing no rooks on the board; and we cannot place 5 or more non-attacking rooks on the board at once. The rook polynomial also gives this information: the coefficient of x^0 is 1 (the constant term) and the coefficients of x^d, for $d \geq 5$ are all 0.

<div align="right">AJD October 2009</div>

4 Maple TA assignment

Now log in to Maple TA and answer questions 1 to 6 of the Rook Polynomial assignment. **You must complete this during the hour of the computer practical. Before you exit from Maple TA you should click "Grade" until you see a message saying your assignment is complete. Then you should "Quit & Save".**

4. For Question 4 the board is:

5. To solve this question you can use the rookpolynomial command as in the previous question. First you need a suitable board. Create a 5×5 board. Label the columns A...E and the rows a...e. Then shade the appropriate squares for the question.

6. This question can be solved in two stages. First sit the women at the table and calculate how many ways this can be done. Then calculate the number of ways the men can sit at the table. To do this label the table like this:
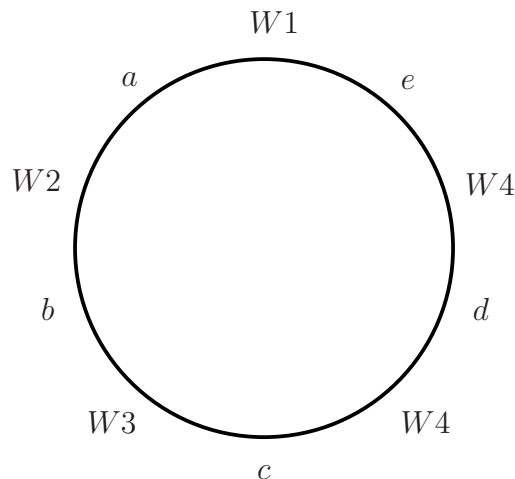
$W1, \ldots, W5$ label places where the women sit and $a, ..., e$ are the possible places where the men, $M1, ..., M5$, can sit. Now create a suitable board and use the rookpolynomial command. To get the total number of positions multiply both of your answers together.